Canada. Marine Sciences Branch.
Manuscript report series.
1-4 1964-67

593°

Manuscript Report Series

No. 1 -4

On the Oceanography of Hudson Bay,
an atlas presentation of data obtained in 1961

F. G. Barber, C. J. Glennie

Programmed by the Canadian Committee on Oceanography

1964 -67

ON THE OCEANOGRAPHY OF HUDSON BAY:

an atlas presentation of data obtained in 1961

F. G. Barber and C. J. Glennie

The Data

The oceanographic data presented pictorially here were obtained on a cruise of the motor vessels "Calanus" and "Theta" into the region of Hudson Bay (Figure 1) during the navigation season of 1961, and were reported in data records of the Canadian Oceanographic Data Centre (1964a; 1964b). The atlas consists mainly of horizontal distributions of temperature, salinity, and dissolved oxygen at standard depths, based on the material in these records. Figure captions are tabulated in List A, and in Figures 3 to 7 is indicated the approximate position of each of the stations occupied. The "Calanus" observations (Figure 3) extended into Roes Welcome Sound and Frozen Strait, with most of the Hudson Bay stations being occupied in late July and early August (List B). These are presented with those occupied during Phase I of the "Theta" cruise. This cruise was carried out from July 22 to October 10, which period was sub-divided for logistic purposes into four phases. This sub-division was adhered to in the data records, and as may be noted in them and in List C here, the relative extent of the serial data obtained during Phases III and IV is quite limited. For this reason, fewer distributions for these Phases are presented, and at some depths the data have been combined.

Preparation

In the preparation of the material, values at standard depth were obtained from a subjective evaluation of the vertical distributions of each property, and from an assessment of the T-S relations. Because of the general shallowness, this procedure was relatively uncomplicated and straightforward, except where the tabulated data are in doubt, for any reason including sampling error. The values were then plotted on the base sheet, and contoured to the appropriate interval. Contour lines are generally solid, but open contours are used occasionally. A dashed contour is one additional to the indicated interval, and a dotted contour indicates that the configuration is in doubt, because either the observation or the interpretation is in doubt. The density of the stations is not high, so that the precision of the contouring cannot be high. Further, the interpretations are not based on complete understanding of the area or the processes operative there, so that the detail of the interpretations will likely undergo revision.

Certain characteristics of the structure of water property appear similar to those observed and described in areas other than Hudson Bay. For example, in August a seasonal thermocline with an associated mixed layer depth existed over much of Hudson Bay. This is shown in the presentation of Figures 85 and 86, derived from an assessment* of the "Theta" bathythermograms of Phase II. The seasonal thermocline is the only significant non-isothermal vertical gradient, so that below the thermocline the structure was generally isothermal at a value about -1.0 to -1.5° C. At some stations a slight minimum temperature occurred below the thermocline at a depth associated apparently with that of the vertical salinity gradient or halocline. This was slight in the north and west where surface and bottom values were about 32.5°/oo, and marked in the south where surface values were less than 30°/oo. The intensity and distribution of the halocline appears to be associated with in-flow of fresh water from land drainage, with vertical mixing due to tidal currents, with the general bathymetry, and with melt-water from ice-cover. **

Some evidence concerning the movement of the ice-cover after break-up suggests that late-season ice frequently is accumulated in the southwest to melt there, as occurred in the 1961 season (Figure 1). Masters of Department of Transport vessels have indicated that ice conditions in Hudson Bay south of latitude 56° and in James Bay were, at the end of July, 1961, the most severe experienced for a number of years. It seemed possible that a portion would survive the summer, but the evidence is that by early September a significant amount of ice did not exist in Hudson Bay, in agreement with the remark of Dunbar (1954, p. 15) that "- - Hudson Bay and Strait become virtually clear even in the worst years".

Remarks Concerning Figures

The plotting sheet, number S-176, was prepared by the Canadian Hydrographic Service for use as the base chart, and is a section of their chart number 5000 titled "Hudson Bay and Strait".

---

* A less subjective analysis utilizing a technique described by Pattullo (1952) has been applied to all the "Theta" bathythermograms. From this it appears that an increase in depth of the mixed layer late in the season in most areas is inhibited, probably by the shallow halocline.

** Hudson Bay is almost completely ice-covered in winter (Hare, F.K. and Montgomery, M.R. 1949; Lamont, A.H. 1949).

The depth chart (Figure 2) is an interpretation of information on Canadian Hydrographic Service charts number 5449 and 5003, derived mainly from reconnaissance survey. Charts to larger scale based on controlled survey are available for a number of inshore areas, as indicated in the Labrador and Hudson Bay Pilot (1954), p. 7.

> "The bay has a greater length of about 930 miles between its extreme latitudes and a greatest width of 520 miles in latitude 60° N. The area of Hudson Bay inclusive of James Bay, Mansel, Coats and the other smaller islands and inlets is 246,000 square miles".

In each vessel, the location of station positions as indicated in Figures 3 to 7 was, to a degree, determined by the requirements of other observations, and had been established prior to the cruise. In "Calanus", other observations included nutrient and productivity measurements, and biological sampling with various types of nets and bottom samplers. In "Theta", on each Phase, a number of geological and geophysical observations were made, either while on station or while proceeding between stations. The program for Phases I and II were carried out generally as planned, except in the area eastward of the Belcher Islands and in the southwest. The survey areas of both Phases III and IV resulted partly from the adverse weather conditions experienced, and partly from considerations of the serial data. In particular, the work of Phase IV represents an attempt to obtain information on the distribution of fresh water in the approach to James Bay and north along the east coast of Hudson Bay, and on the extent of changes in the deep water in the northeast.

In the distributions of Phase I, Figures 8 to 34, the location of the "Calanus" observations at each depth is indicated by open circles. These and the "Theta" observations were brought together without undue difficulty, although doubtful data exist in the records for both. Specifically, the tabulated data for "Theta" station 35 clearly exhibit errors. The deck log for the station indicates that the cast was successful, and that all reversing bottles were returned to the rack in proper order. It has been concluded that a blunder was made drawing the salinity samples, but a satisfactory rearrangement of the data for the station has not been possible. As well, the tabulated data for station 120 are believed to be in error at depths greater than 30 metres, apparently due to a post-trip of bottles below that depth.

The presentation of each property at a depth of "deepest observation" is used rather than as a "bottom observation" as no attempt was made during the survey to obtain serial data consistently close to the bottom. Such presentation is made only for the observations of Phases I and II for, as mentioned, the data for the later Phases are limited. For this reason also, it seemed appropriate to combine the material of Phases III and IV, and this has been done for most of the data (Figures 66 to 84), but not for the temperature distributions at the surface and 10 metres.

In Figures 87 and 88 are presented the T-S and T-O_2 relations for the "Theta" serial data by phase number. For each phase, a number of values lie outside the range indicated, particularly in the T-S presentation because of the prevalence of low-salinity water. Two such values were observed during Phase I, thirty-seven during Phase II, and three during Phase IV; the lowest salinity observed was 15.33°/oo at station 96. With regard to dissolved oxygen, two suspiciously low values were observed at station 136 of Phase II, one of which lies outside the range of the figure. The freezing point shown in Figure 87 is based on the work of Thompson (1932) according to LaFond (1951). In the T-O_2 relation of Figure 88 saturation at 27.11 and 32.5°/oo are as found by C. J. J. Fox according to Sverdrup et al. (1942, p. 188).

REFERENCES

Canadian Hydrographic Service,
Dept.Mines & Tech. Surv., Ottawa
1954: Labrador and Hudson Bay Pilot. Queen's Printer Cat. No. M51-1354, Ottawa.

Canadian Oceanographic Data Centre,
Dept. Mines & Tech Surv., Ottawa
1964a: Data Record Series No. 1. Hudson Bay Project, 1961. Queen's Printer Cat. No. 58-1/1964-1.

1964b: Data Record Series No. 12. Hudson Bay Project, 1961. Queen's Printer Cat. No. 58-1/1964-12.

Dunbar, Moira
1954: The pattern of ice distribution in Canadian arctic seas. Roy. Soc. Can., Trans. VXLVIII, ser. III.

Hare, F.K. and Montgomery, M.R.
1949: Ice, open water, and winter climate in the eastern arctic of North America. Arctic, 2:2 ; 79-89. Pt. 1, Sept. 1949; 2:3: 149-164 Pt. 2 Dec. 1949.

Lamont, A.H.
1949: Ice conditions over Hudson Bay and related weather phenomena. Am. Meteorol. Soc., Bull. 30, 8:288-289.

LaFond, E.C.
1951: Processing oceanographic data. U.S. Navy Hydrog. Off., Pub. 614

Pattullo, J.
1952: Mixed layer depth determined from critical gradient frequency. Scripps Inst. Oceanog., Ref. 52-25.

Sverdrup, H.V., Johnson, M.W. and Fleming, R.H.
1942: The oceans; their physics, chemistry and general biology. Prentice-Hall, Inc., New York.

Thompson, T.G.
1932: The physical properties of sea water, in Physics of the Earth, vol. 5, Oceanography, p. 63-94; and Nat. Res. Council, Bull. 85.

LIST A

Figure Captions

Phase II

"Calanus" repeat station 21. "Theta" stations 59-187.

Phase III

"Calanus" repeat station 6. "Theta" stations 188-235.

Phase IV

"Theta" stations 236-265

Phases III and IV

Miscellaneous

LIST B

"CALANUS"

Data and stations occupied at which serial data were observed

Day/Month	Station Nos.	Day/Month	Station Nos.
22, July	2	10 (con't)	26, 27, 28
23	3	11	29, 30, 31
24	4	14	32
25	5	20	33
26	6, 7*	23	34
27	8	24	35, 36, 37
30	10, 11	25	39
31	12, 13	26	40
2, August	14	27	41, 42
6	15, 16, 17, 18	28	43, 44, 45
7	19, 20, 21, 22	31	46
9	24	1, September	21**
10	25 and	10	6**

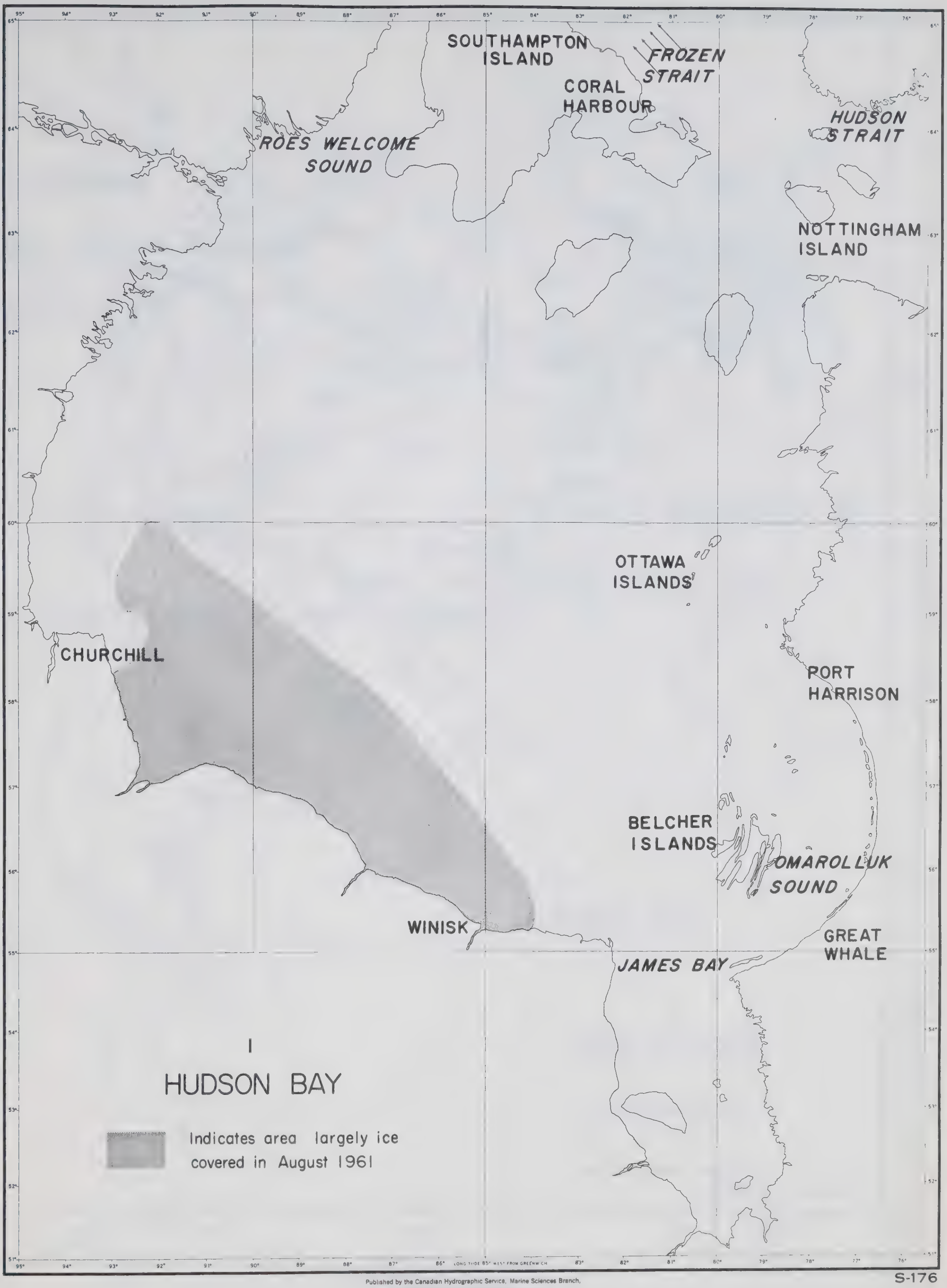
* station 7 was occupied in Knight Harbour, Marble Island.
** a re-occupation of the earlier stations.

LIST C

"THETA"

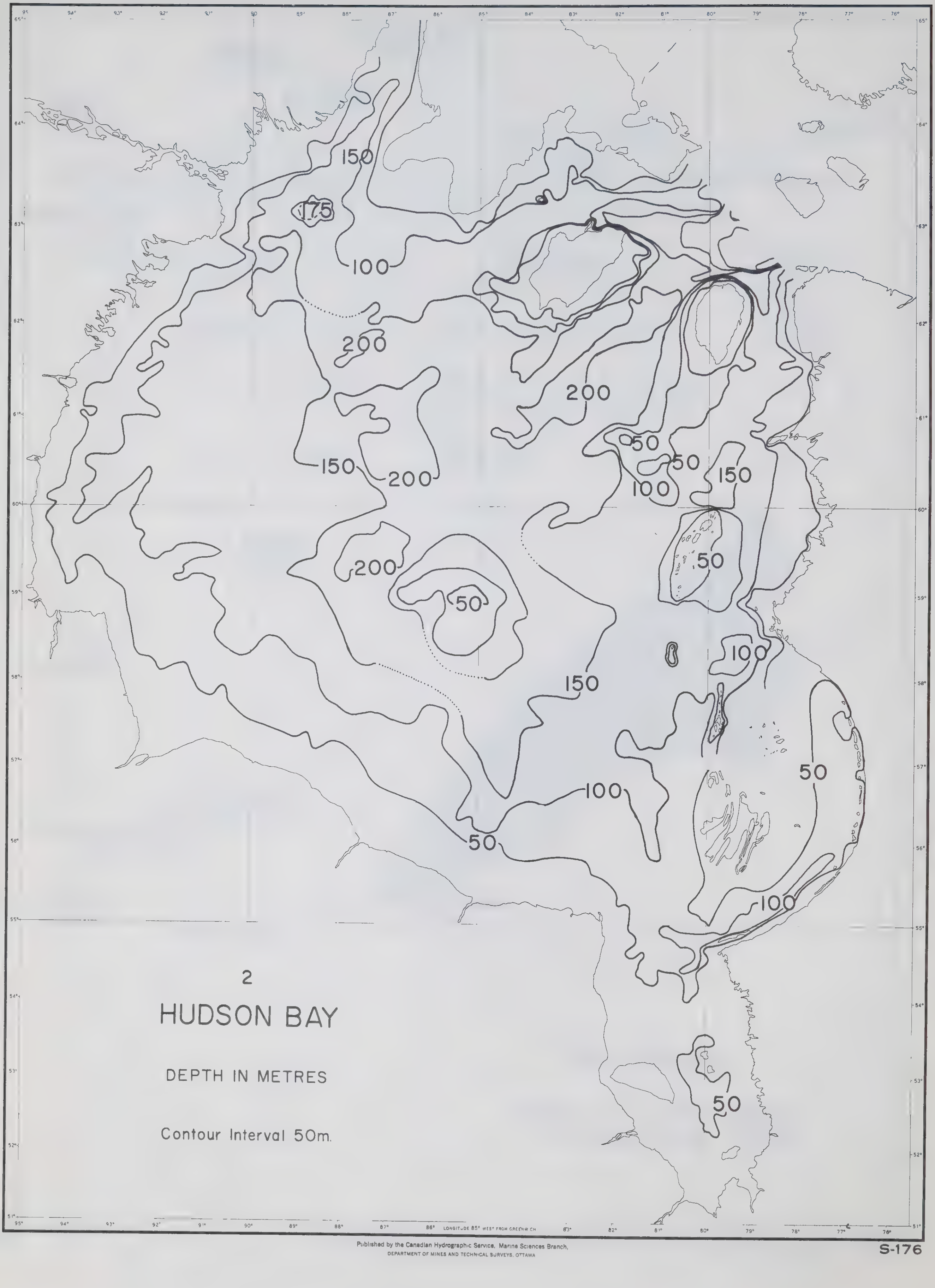
Stations occupied at which serial data were observed, by Phase number and date, station number, and the total number of stations occupied.

Phase No.	Date	Station No.	Total
I	July 22 - 31	1 to 52, 55 to 58	56
II	August 3 - Sept. 3	59 to 126, 128 to 187	128
III	Sept. 10 - Sept. 21	188 to 197, 200, 210, 213, 215, 217, 219, 222 to 227	22
IV	Sept. 27 - Oct. 8	236 to 265	30



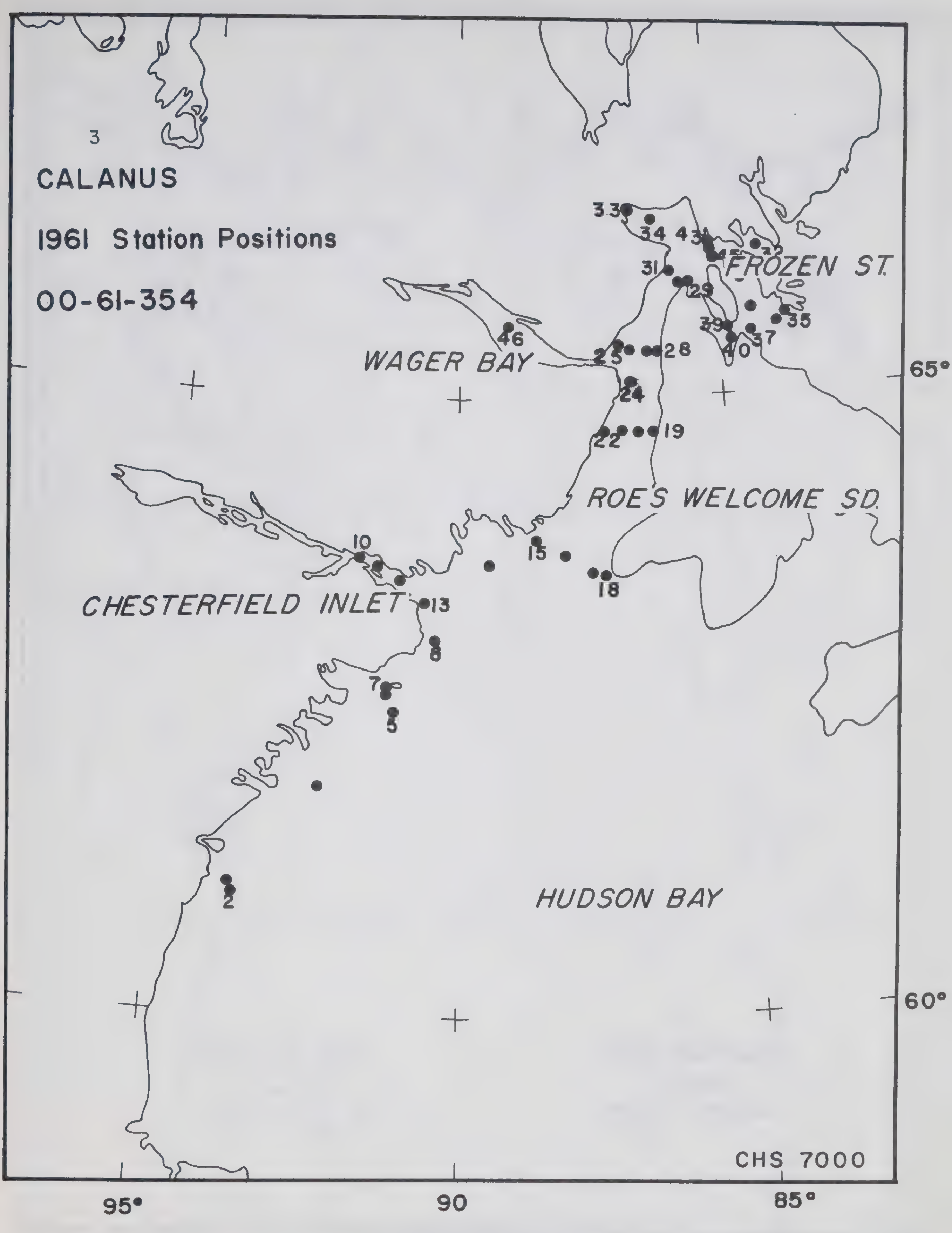
S-176

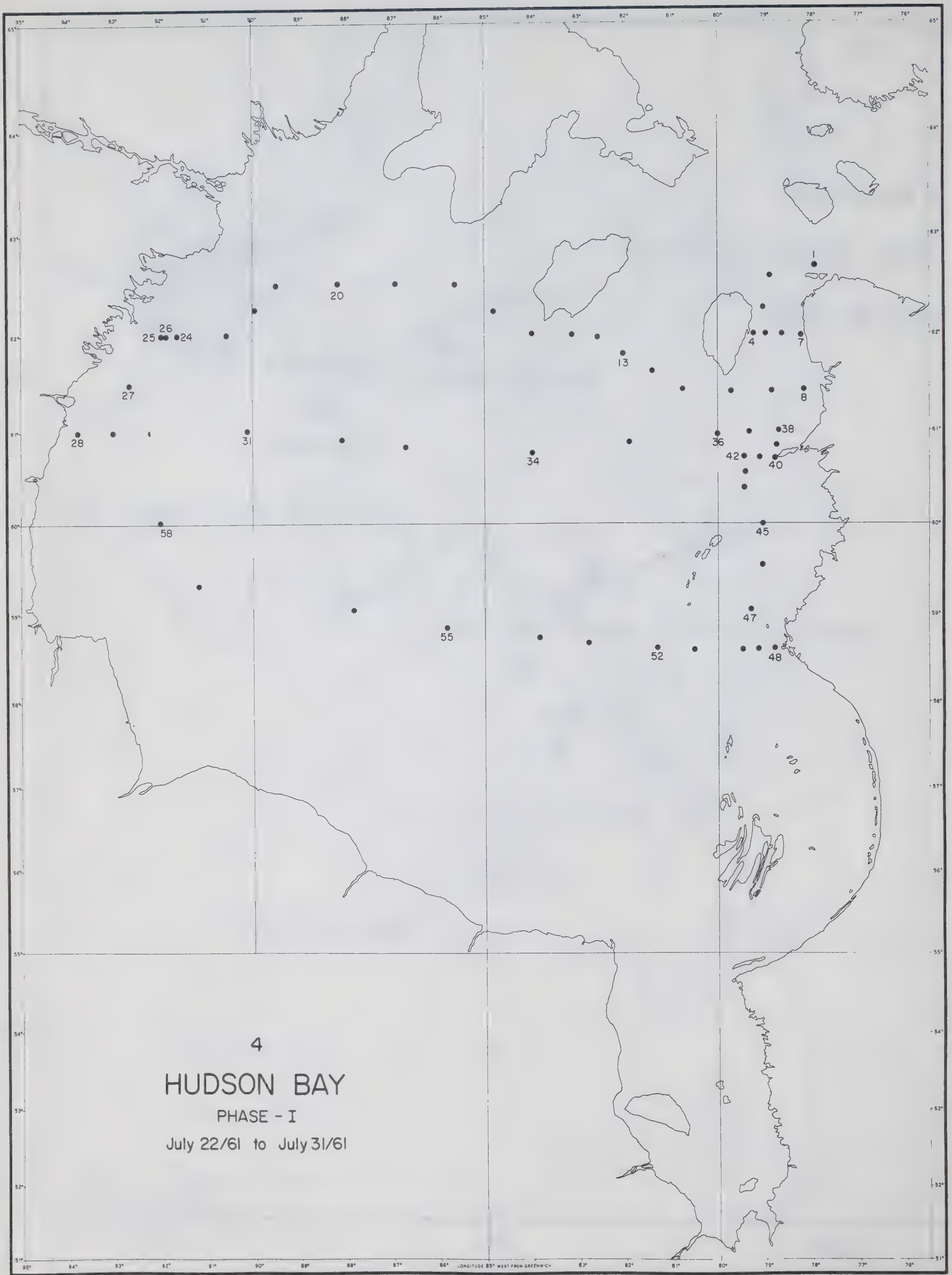
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch, DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

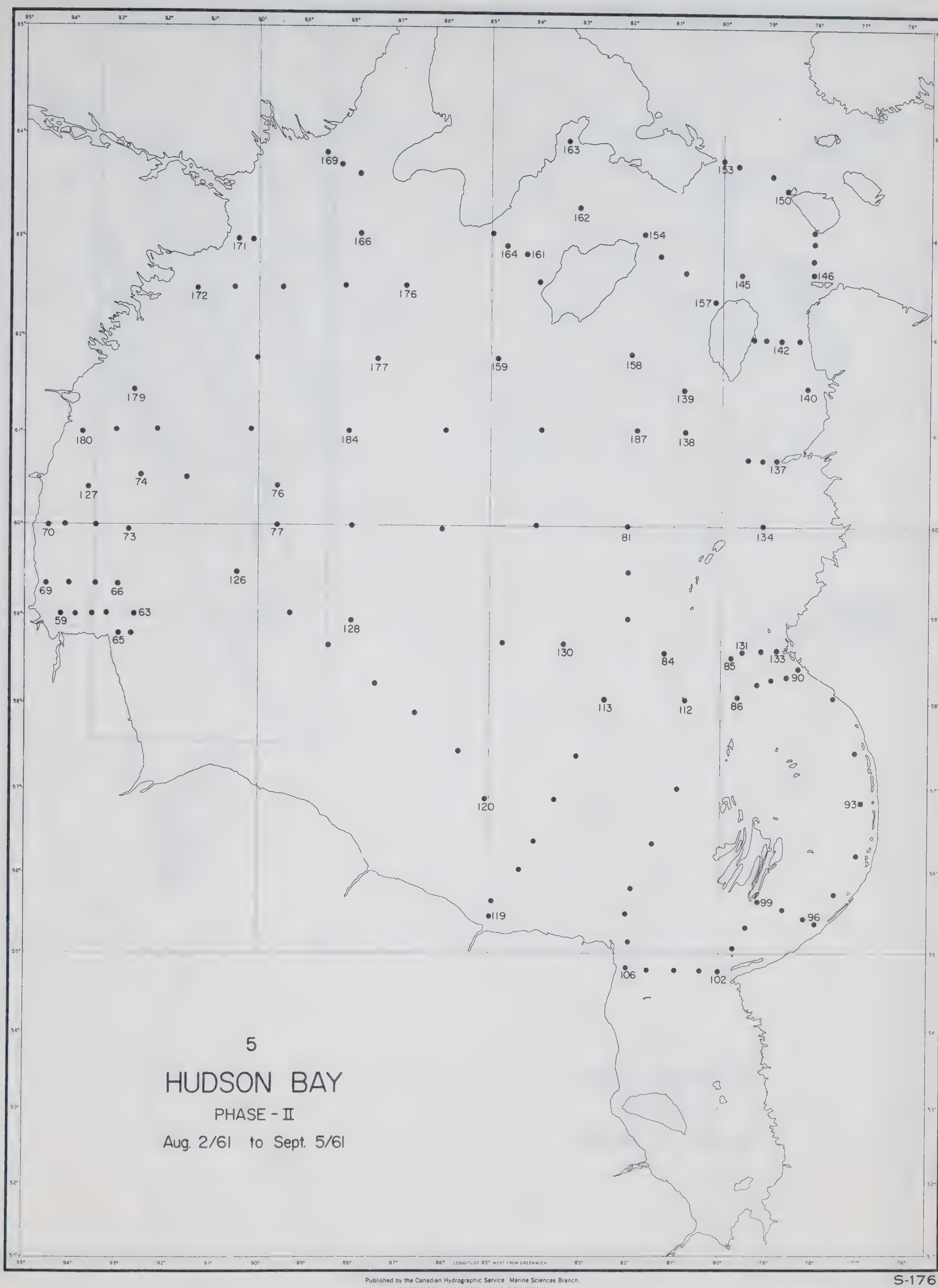
S-176


3
CALANUS
1961 Station Positions
00-61-354
FROZEN ST.
WAGER BAY
ROE'S WELCOME SD.
CHESTERFIELD INLET
HUDSON BAY
CHS 7000
65°
60°
95°
90
85°
33
34
43
31
29
35
39
37
40
46
25
28
24
19
22
10
15
18
13
8
7
5
2



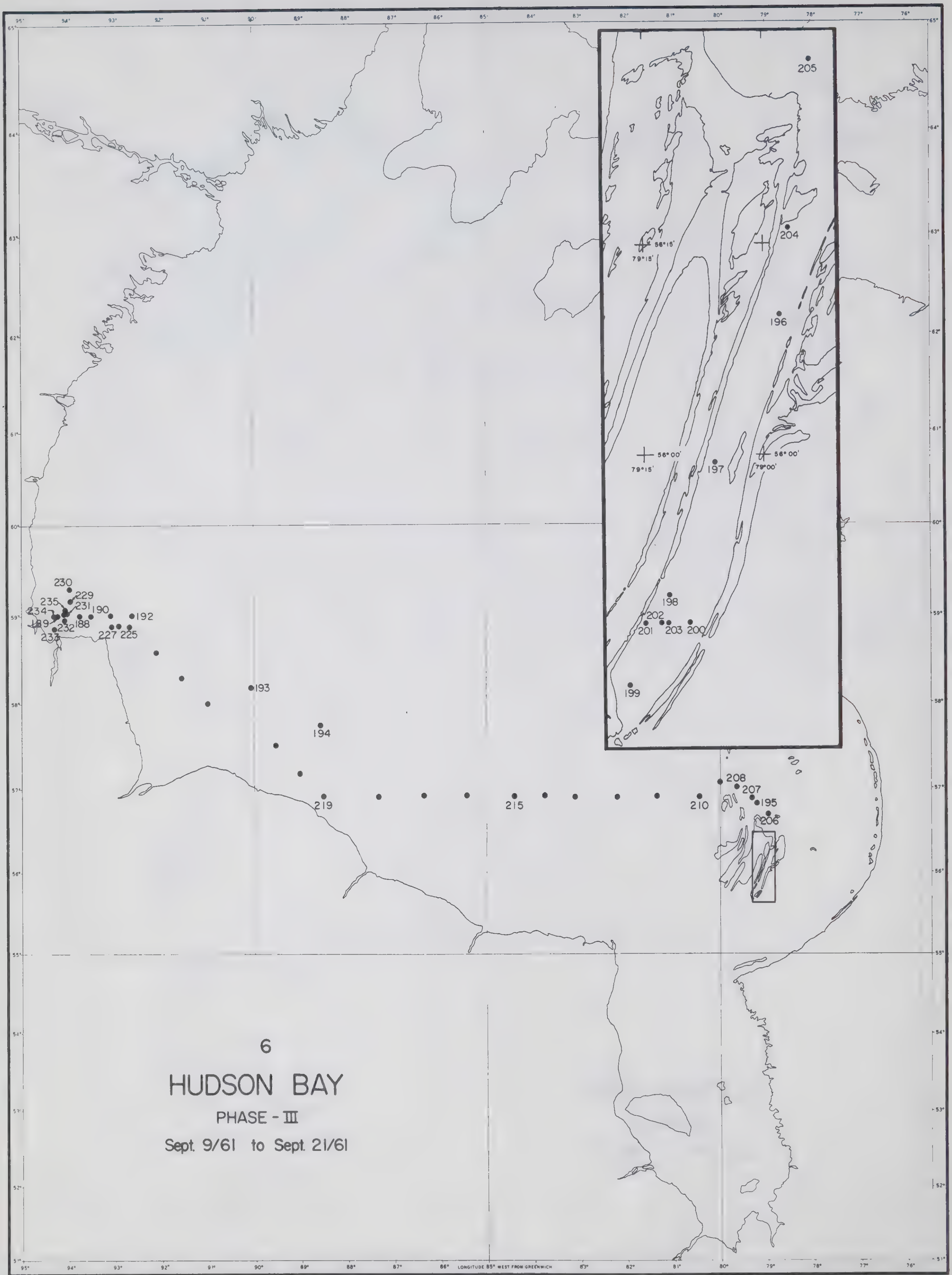
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA

S-176



Published by the Canadian Hydrographic Service Marine Sciences Branch.
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA

S-176



6

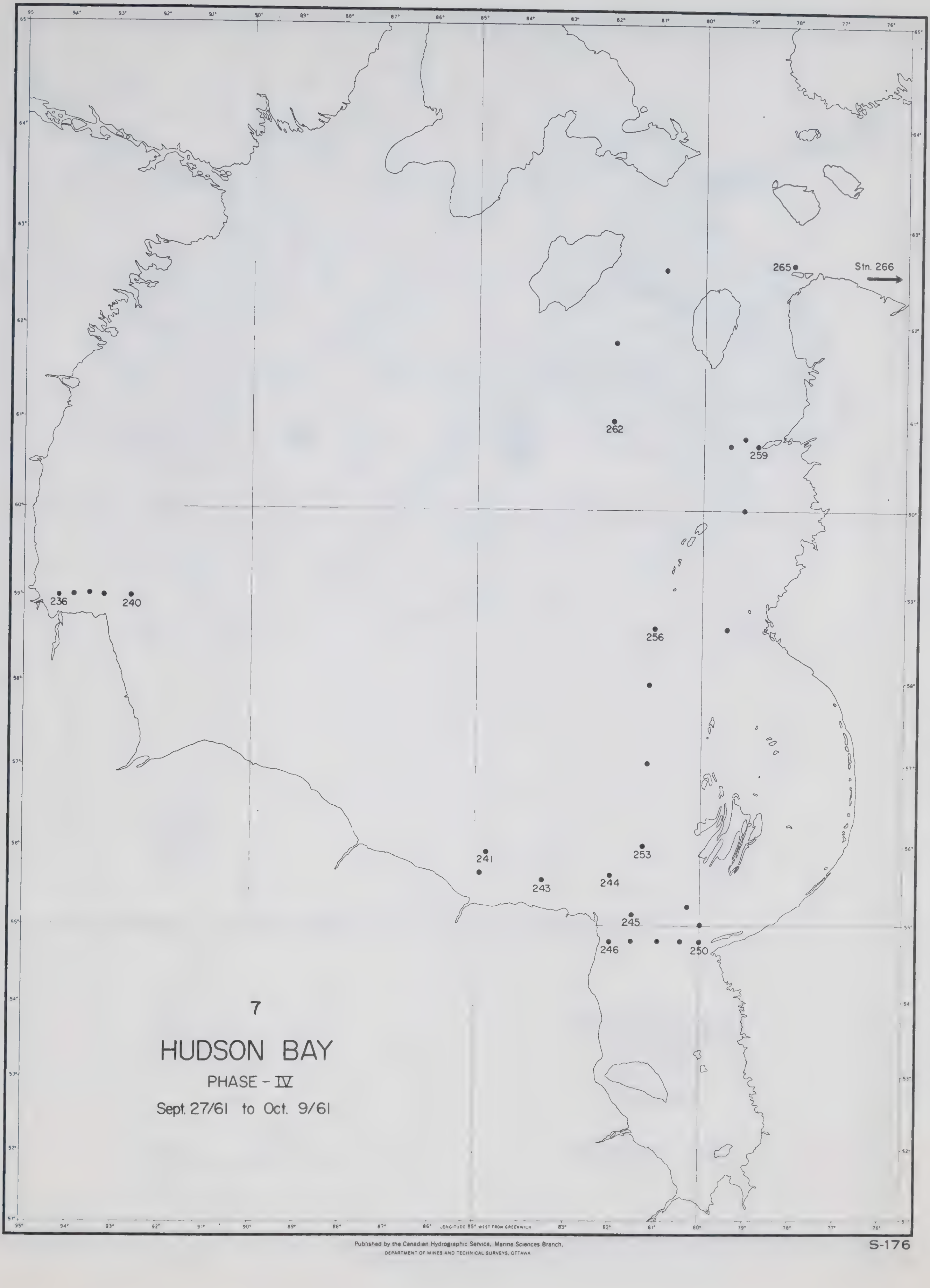
HUDSON BAY

PHASE - III

Sept. 9/61 to Sept. 21/61

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



7

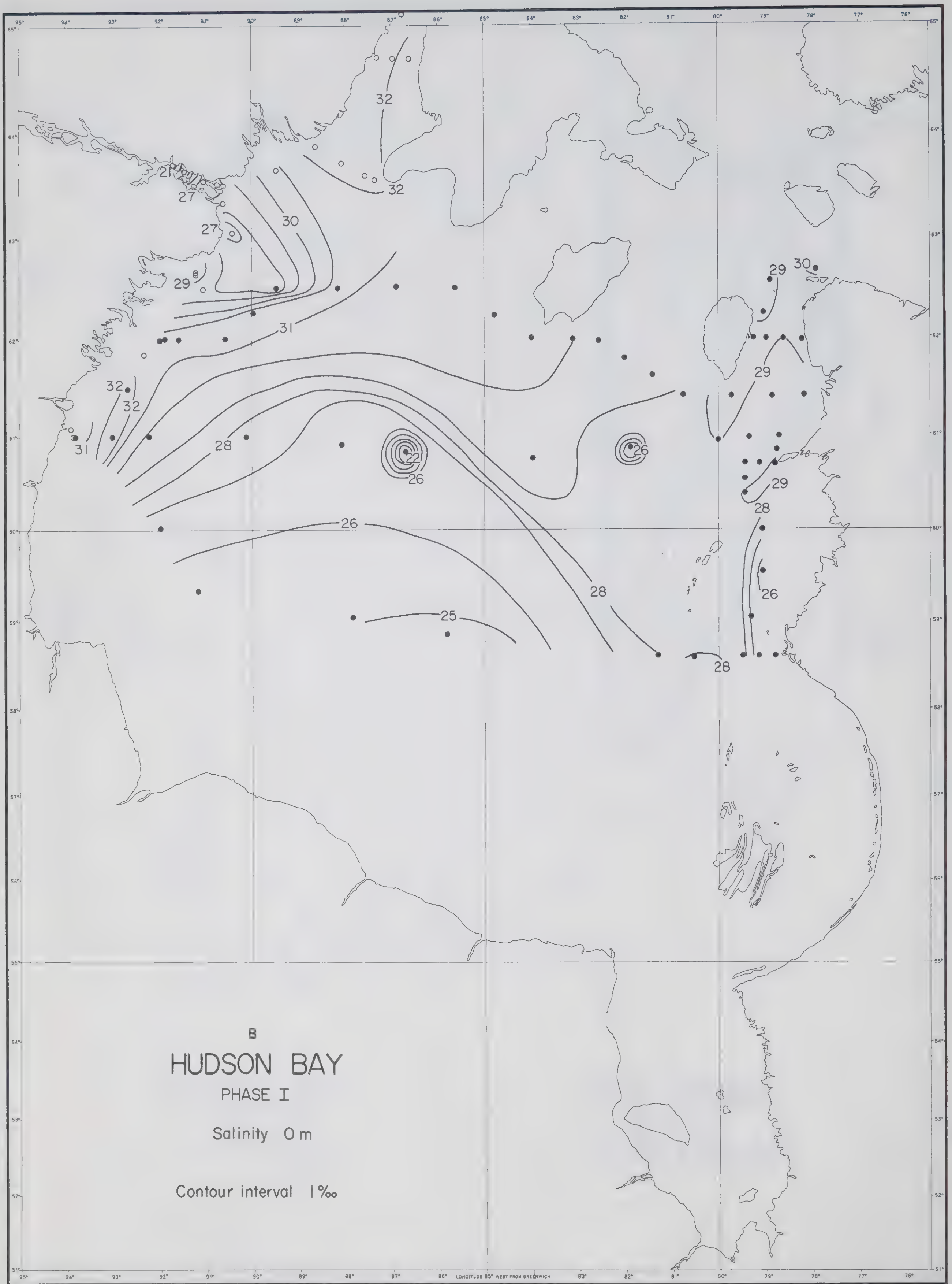
HUDSON BAY

PHASE - IV

Sept. 27/61 to Oct. 9/61

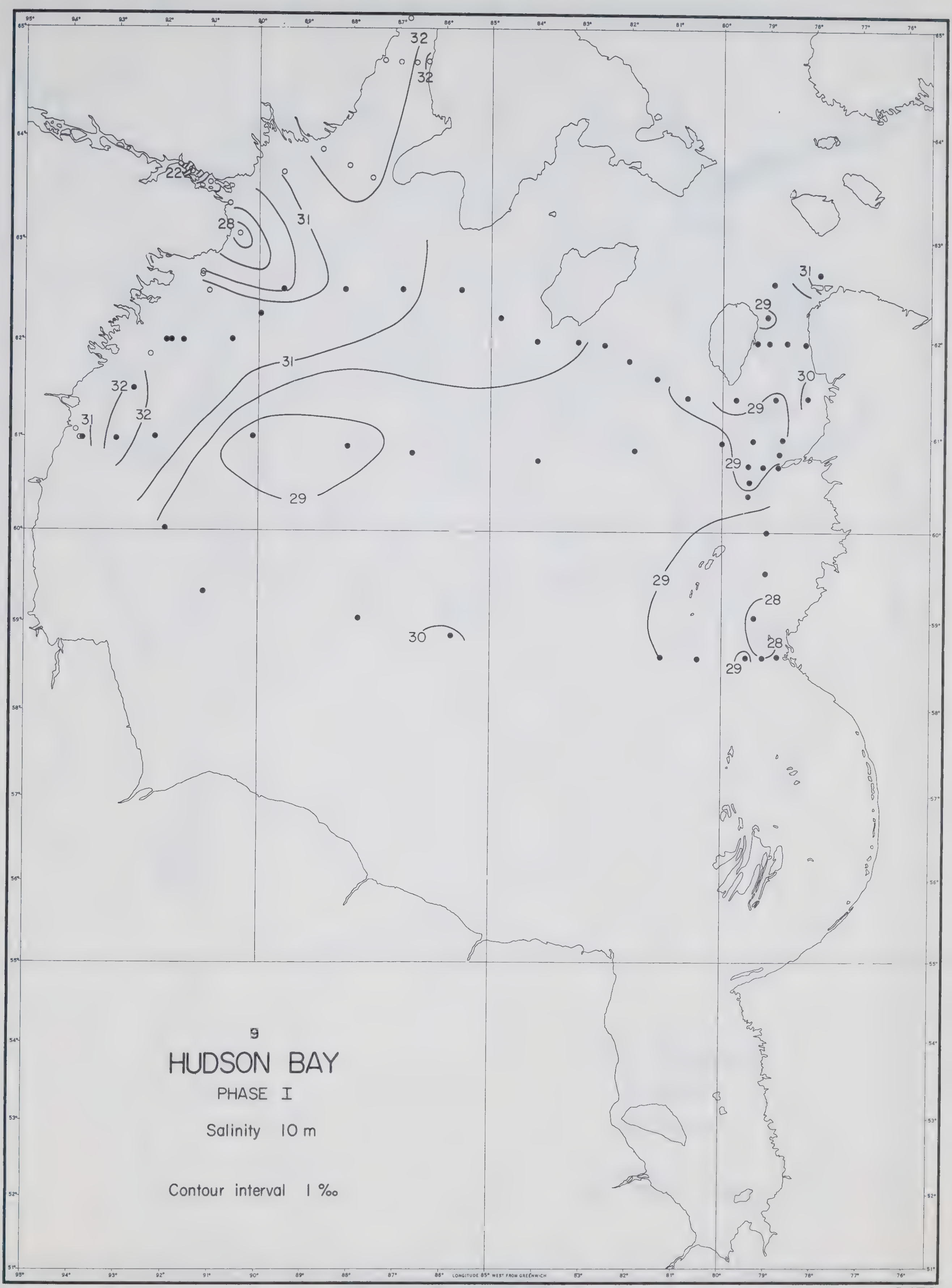
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176

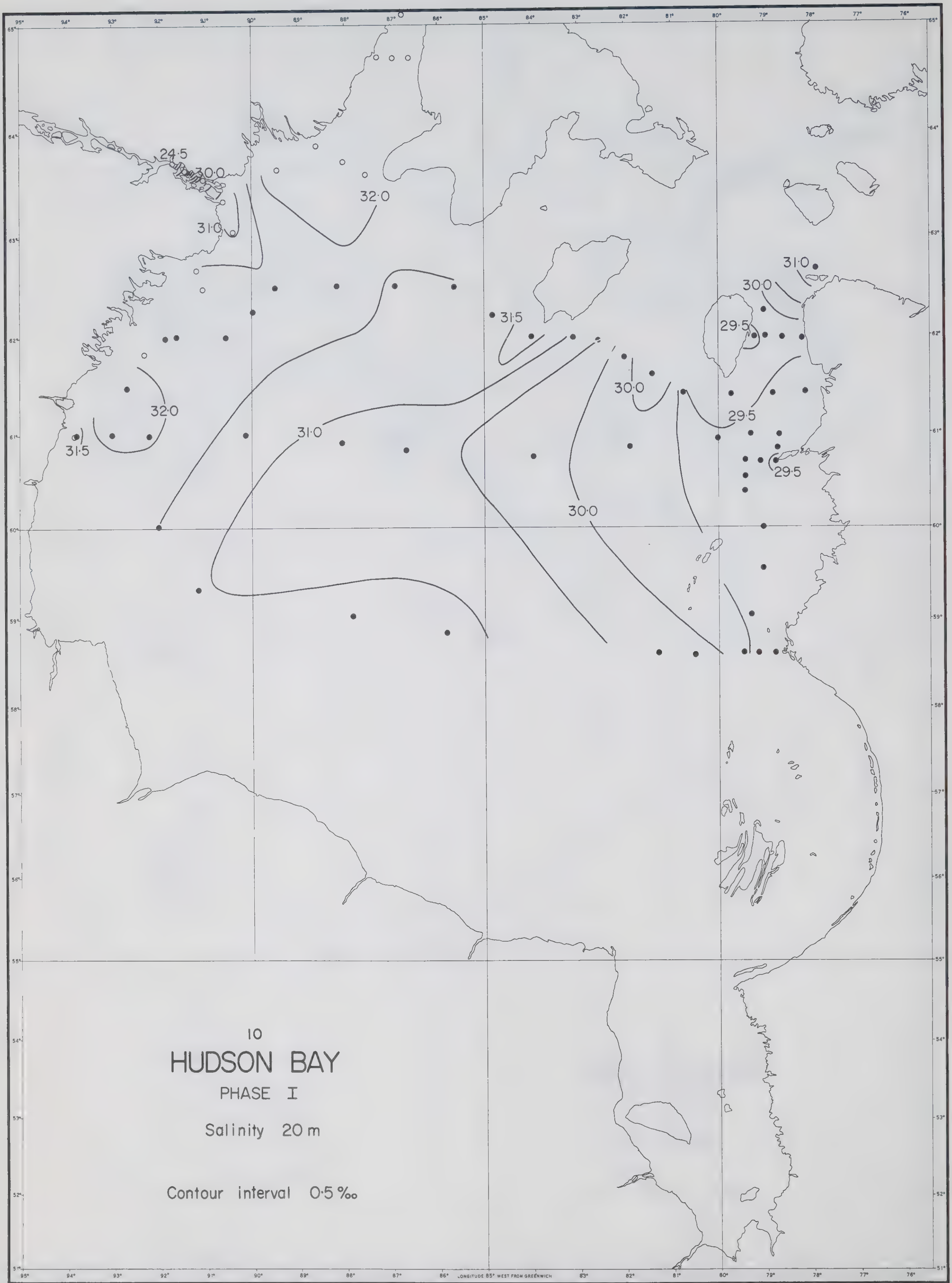


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176

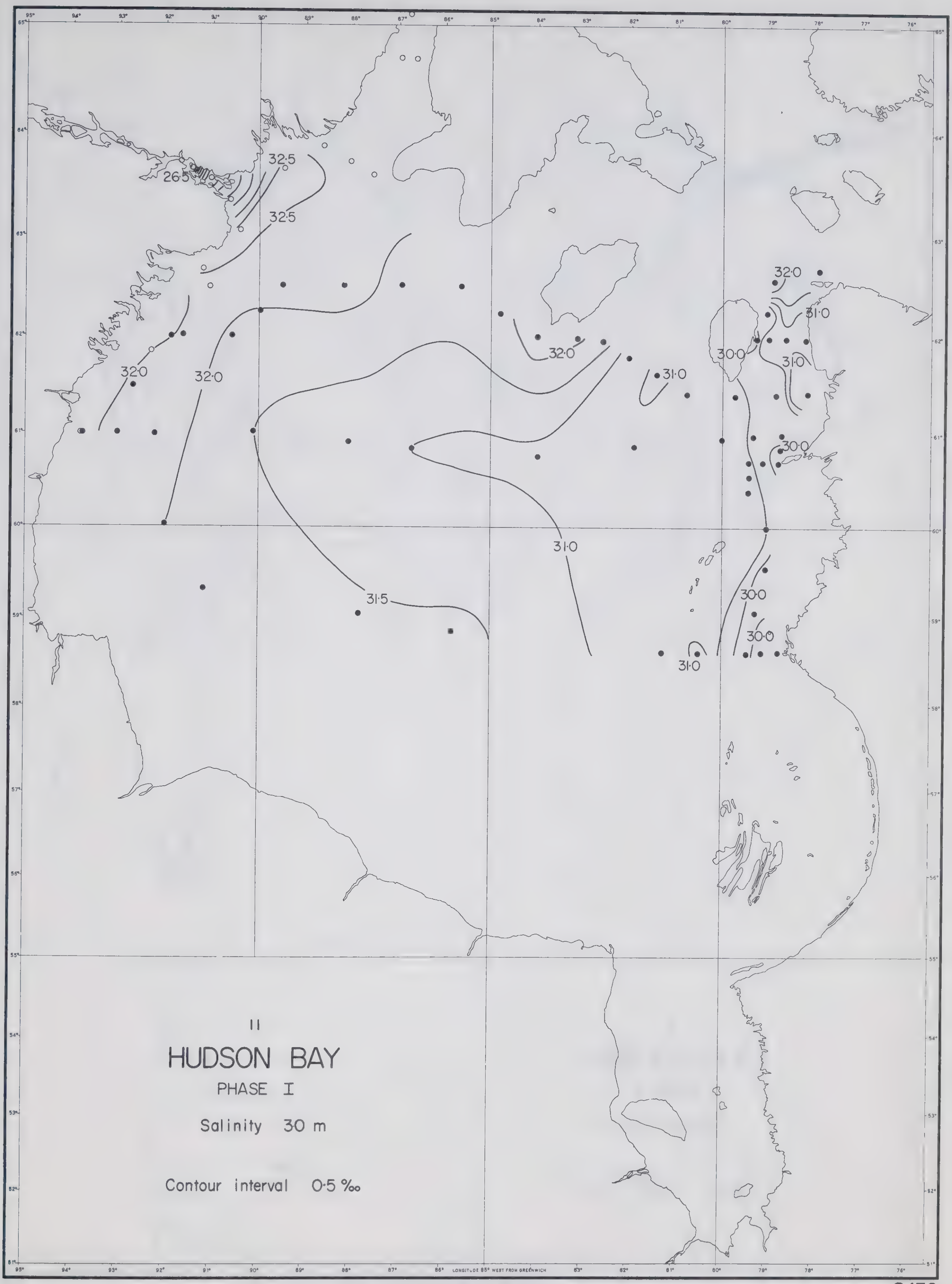


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

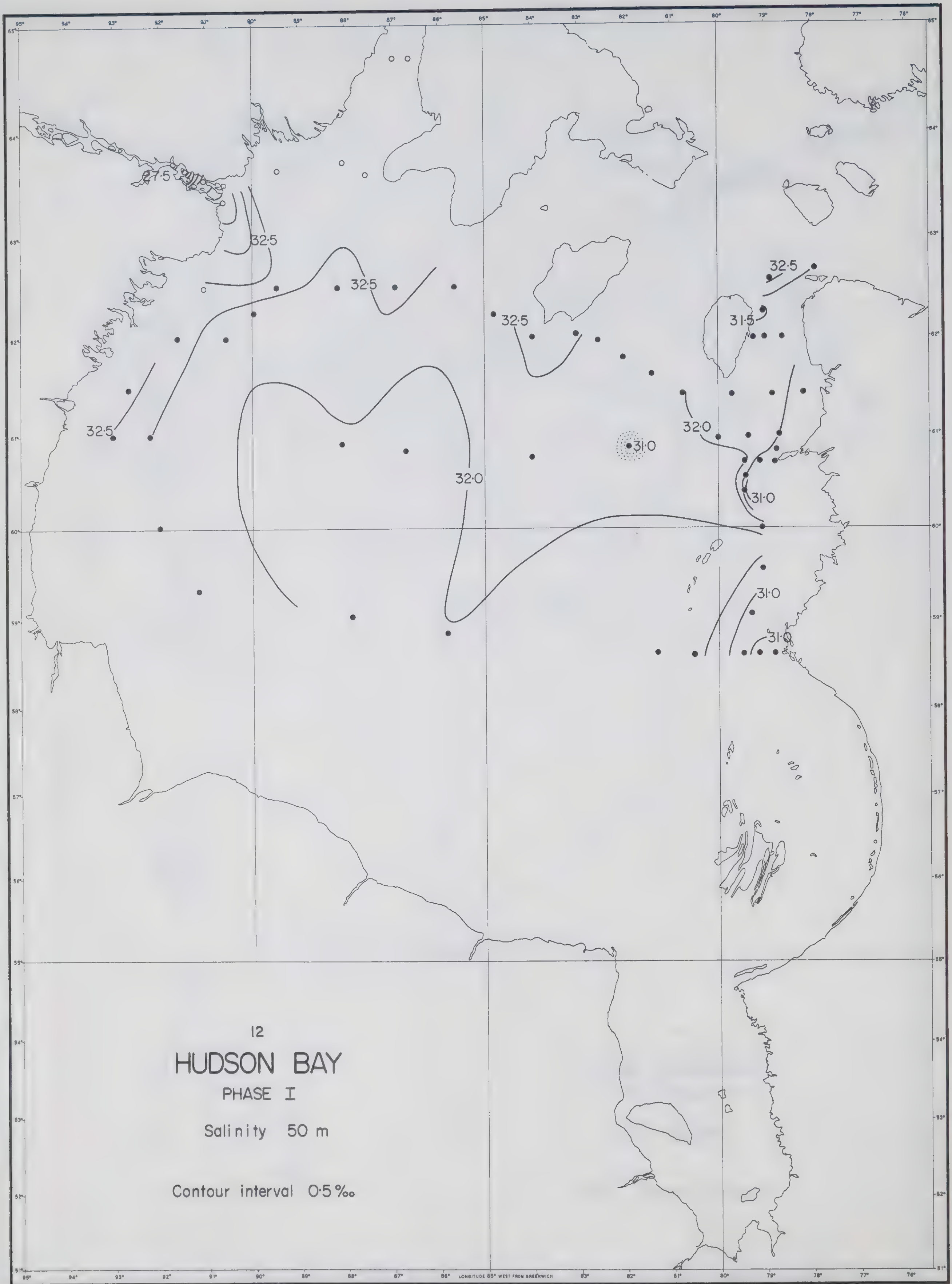


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



12

HUDSON BAY

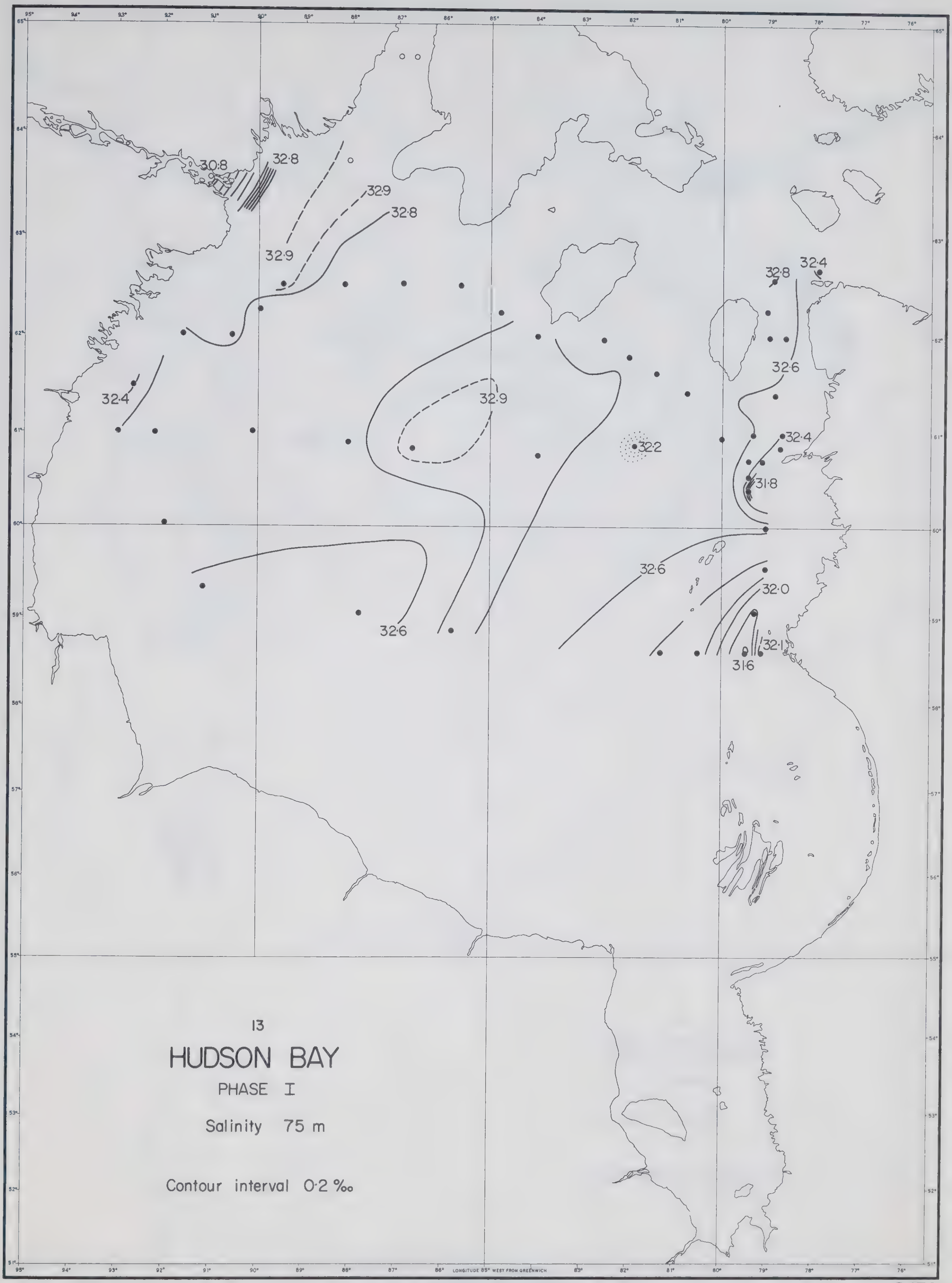
PHASE I

Salinity 50 m

Contour interval 0·5 ‰

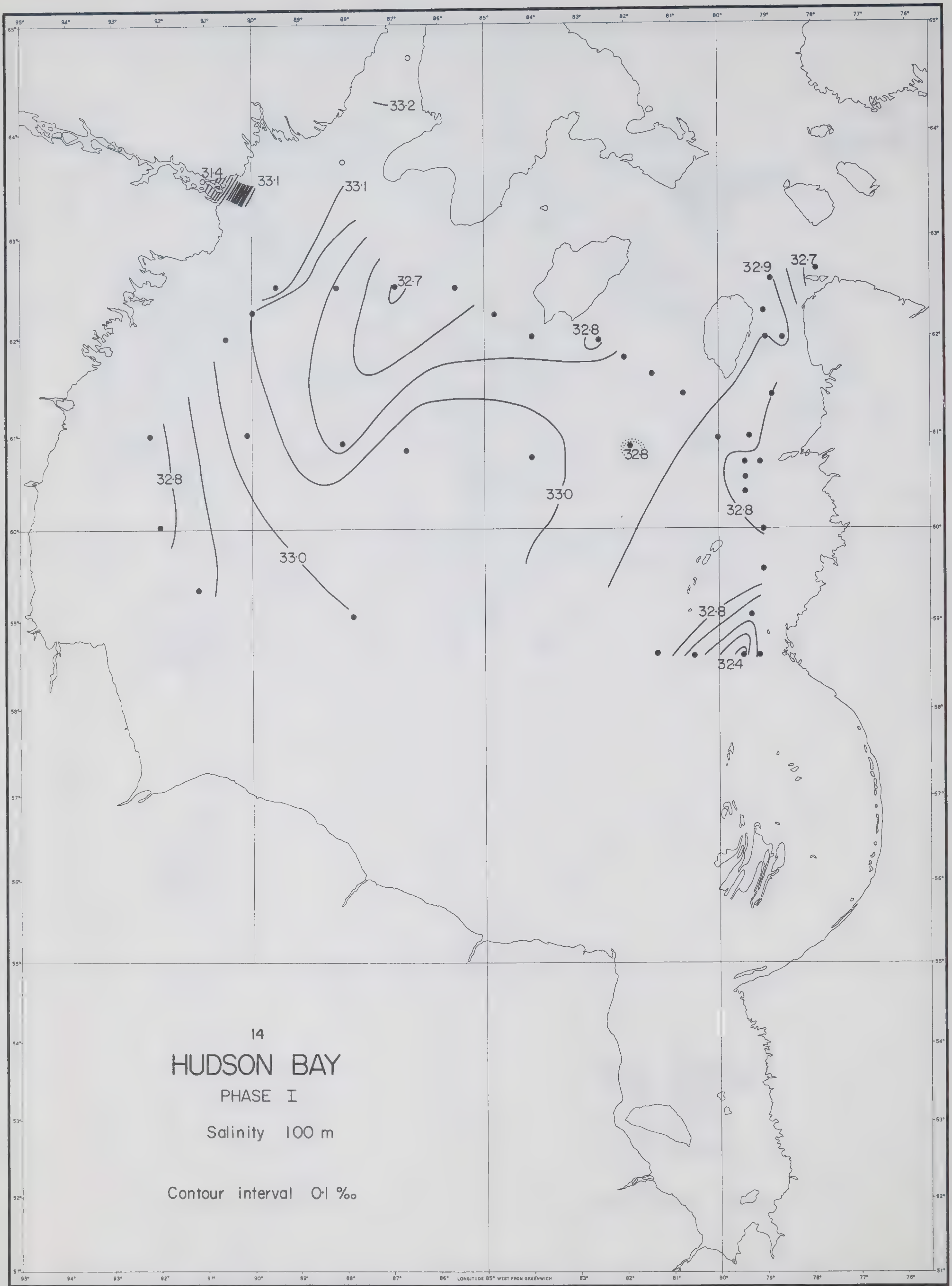
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



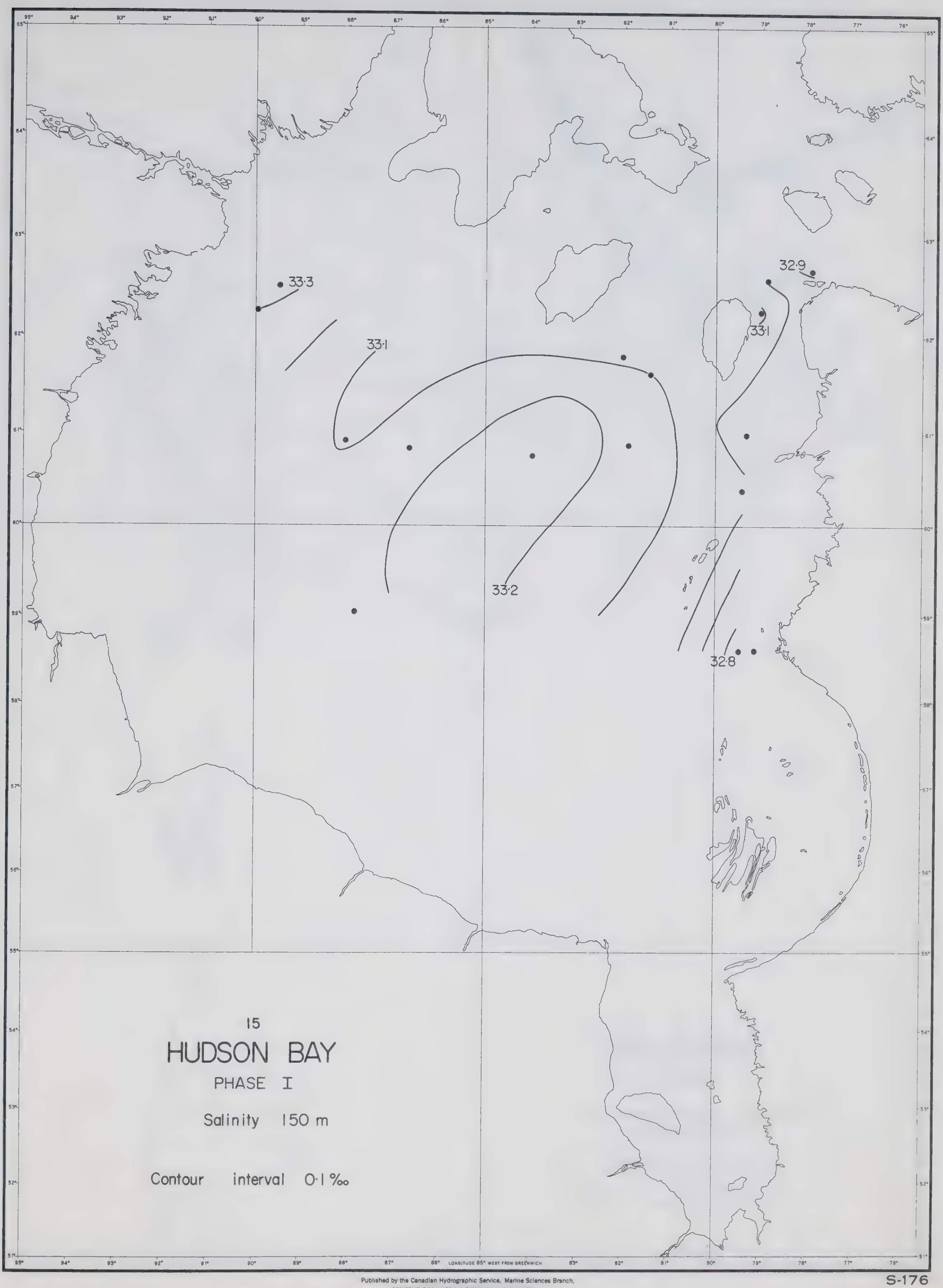
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA

S-176

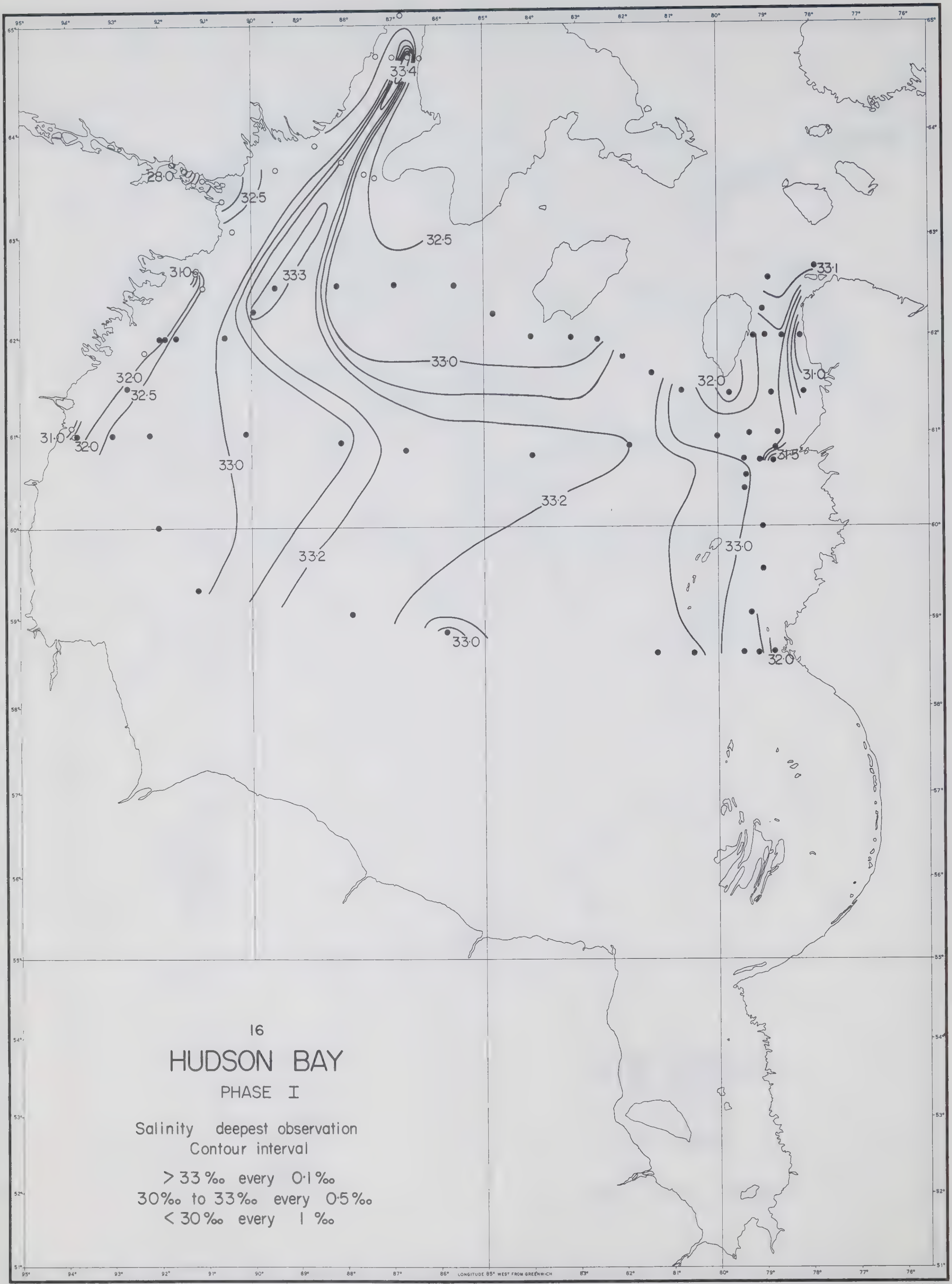


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176

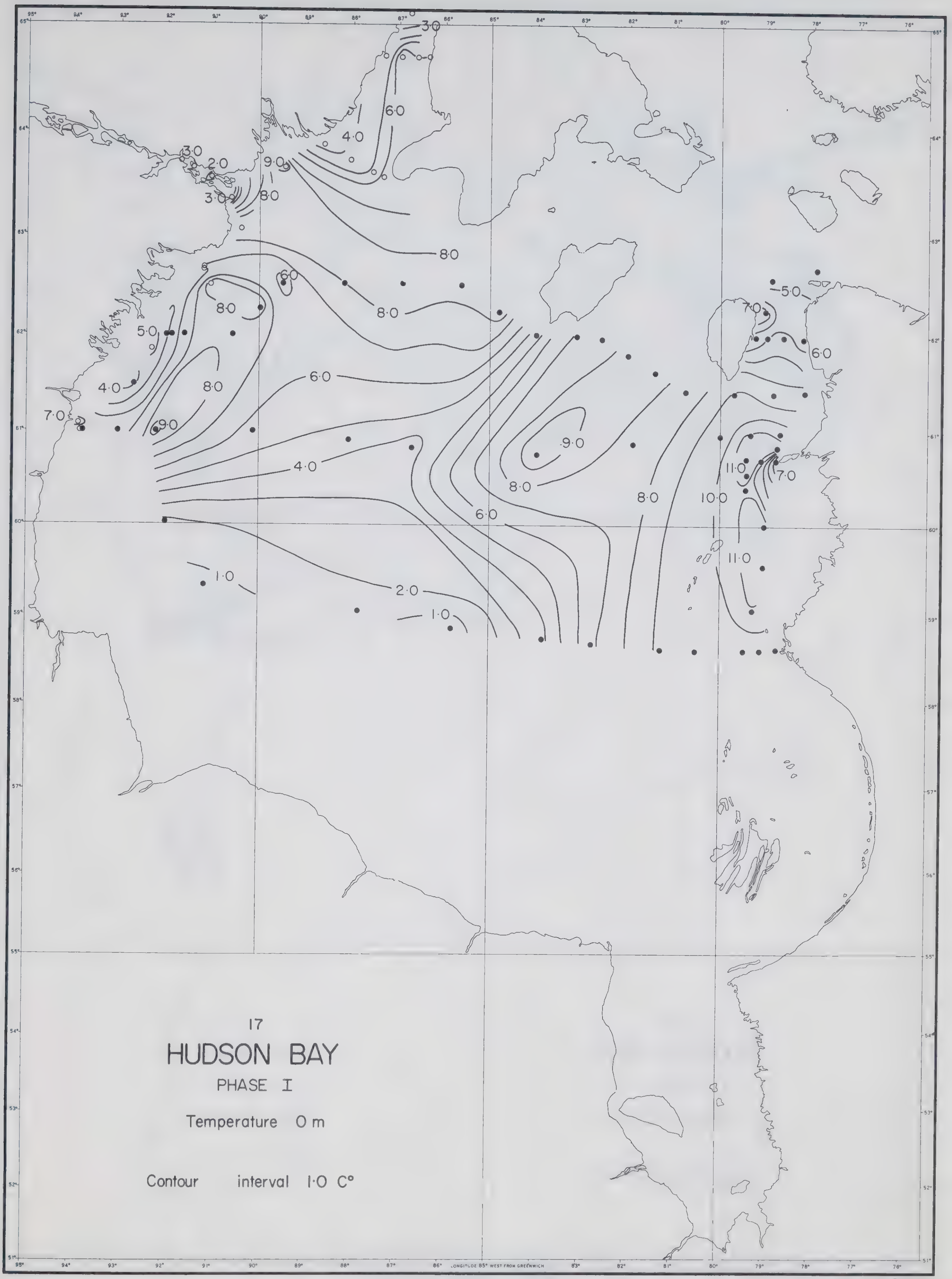


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



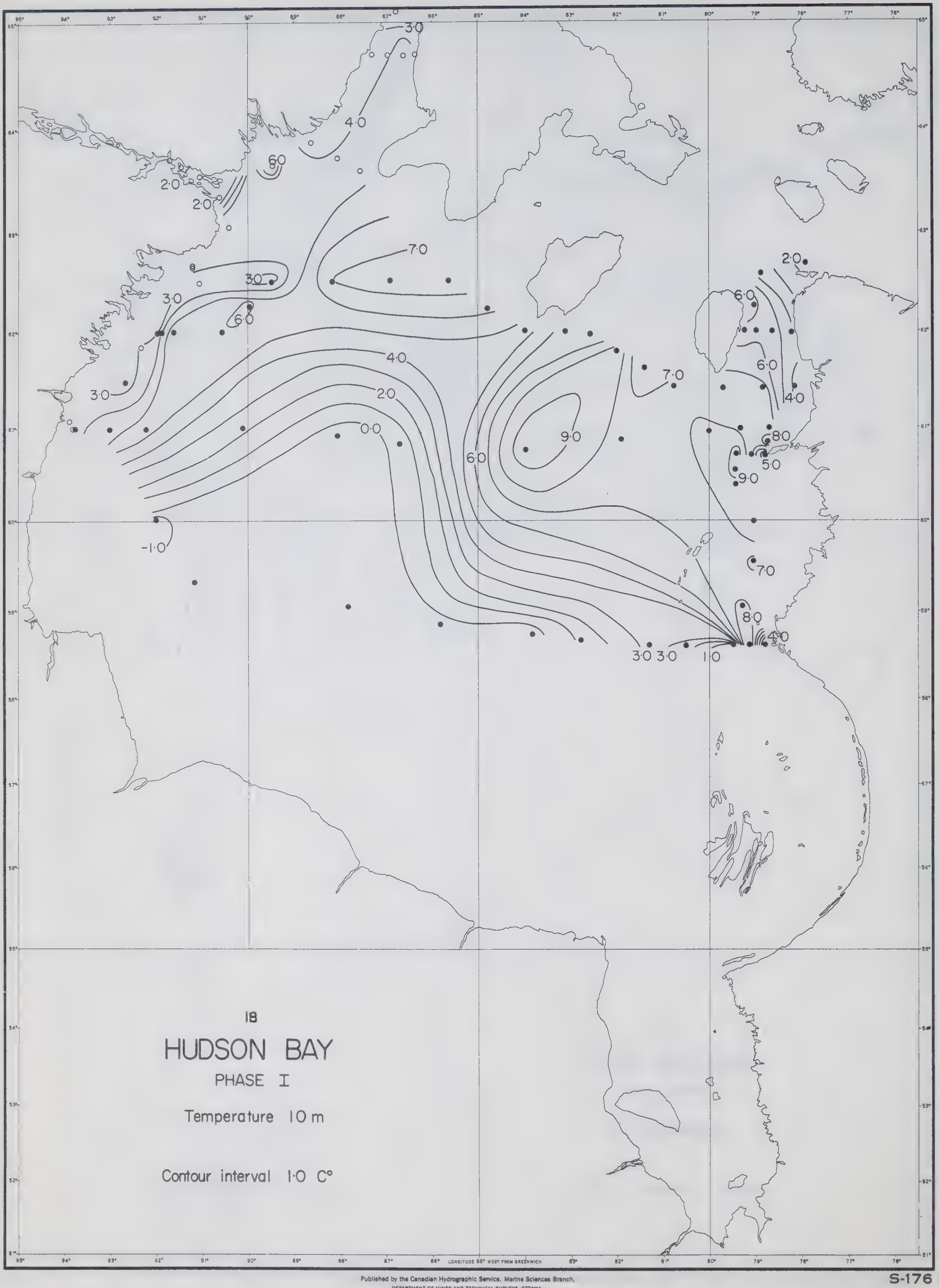
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



18

HUDSON BAY

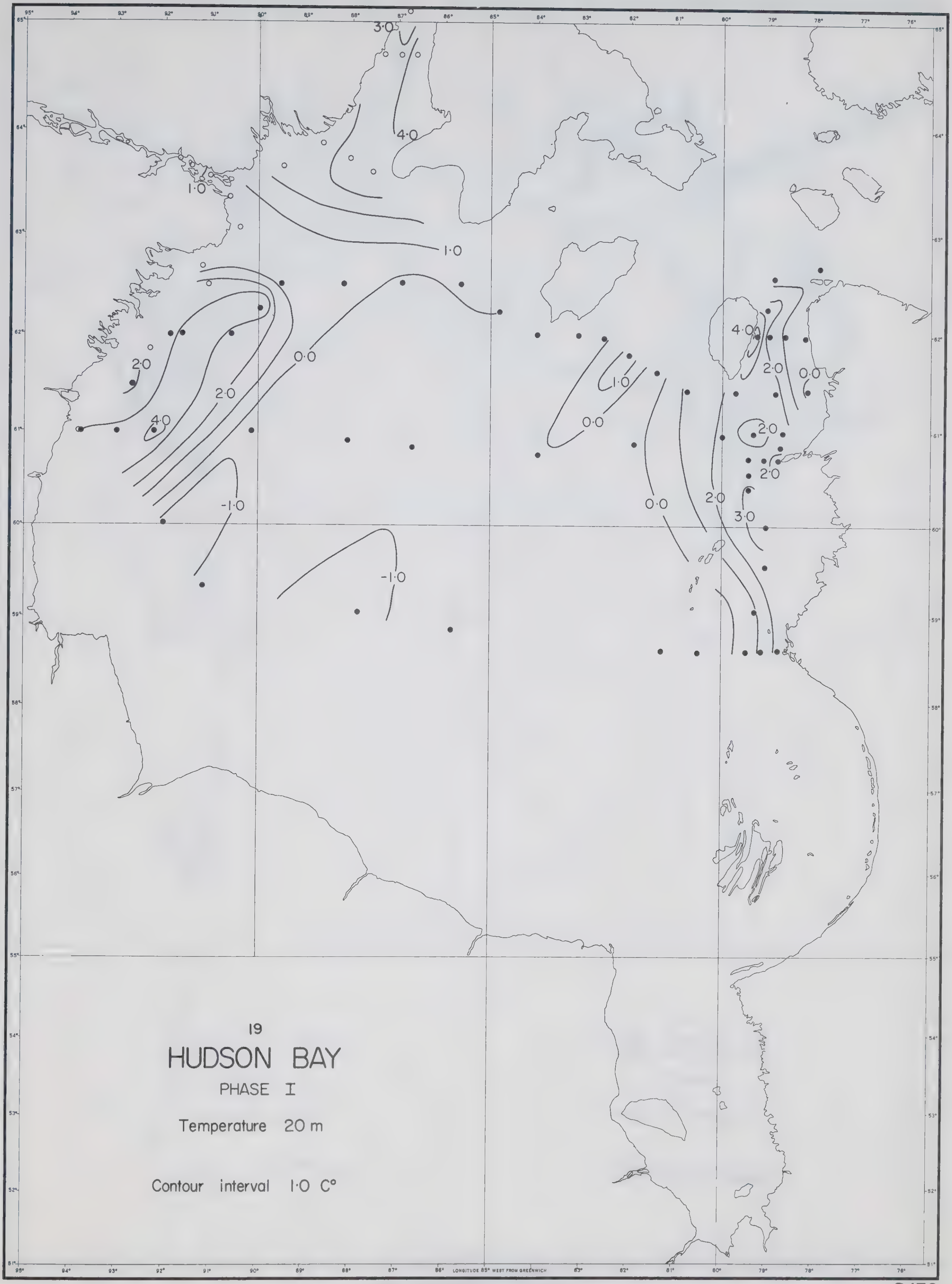
PHASE I

Temperature 10 m

Contour interval 1·0 C°

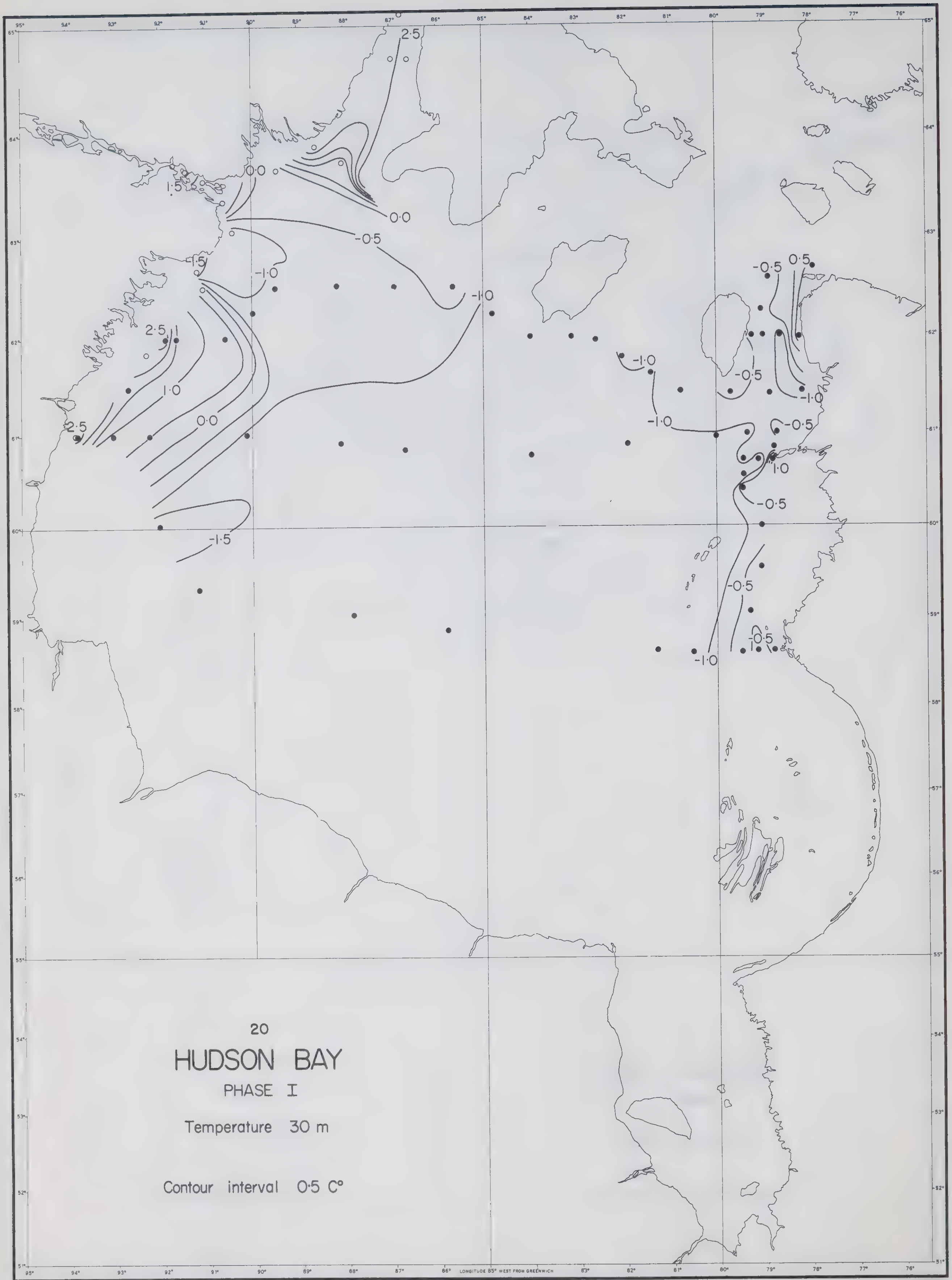
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



20

HUDSON BAY

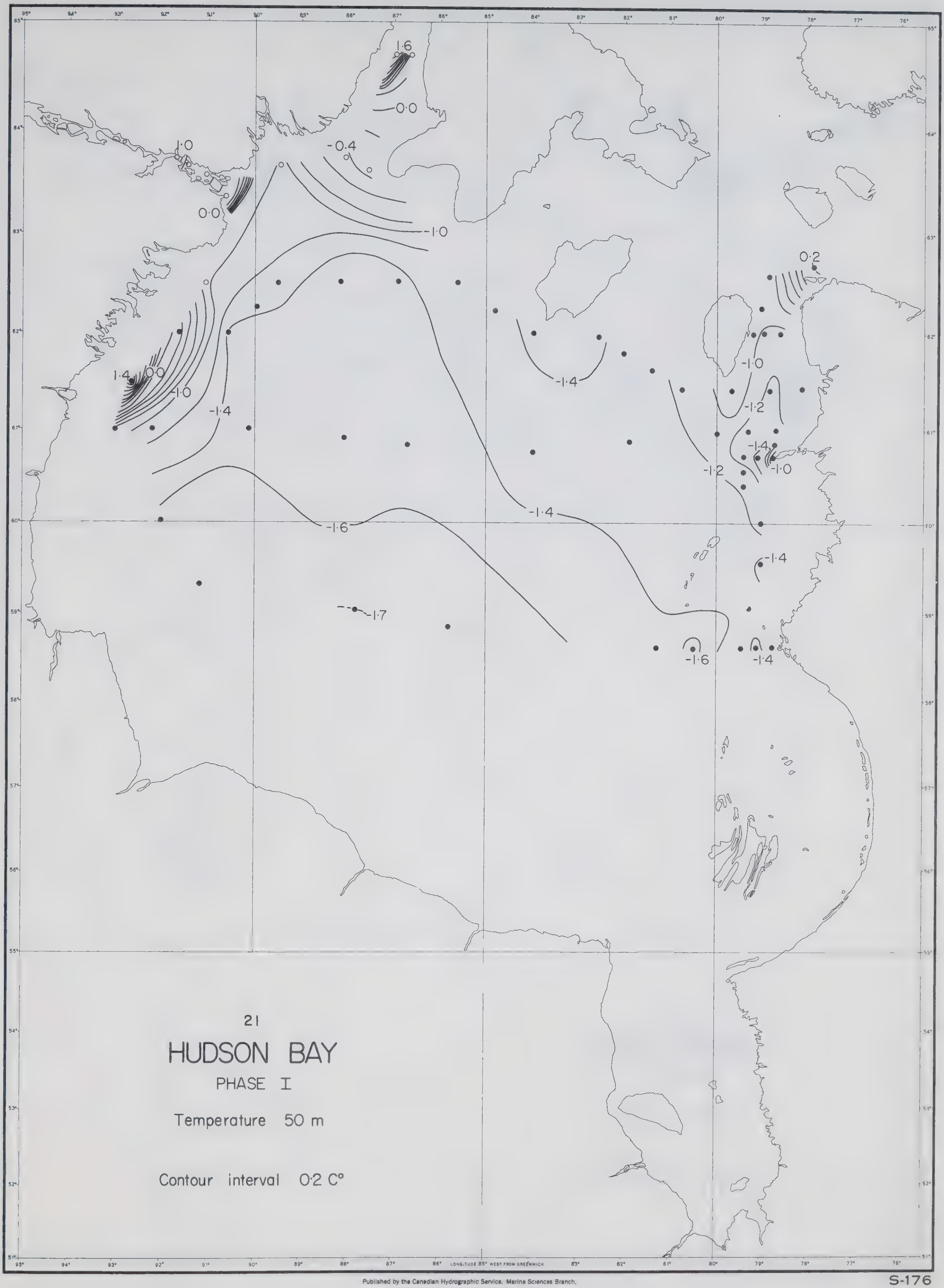
PHASE I

Temperature 30 m

Contour interval 0·5 C°

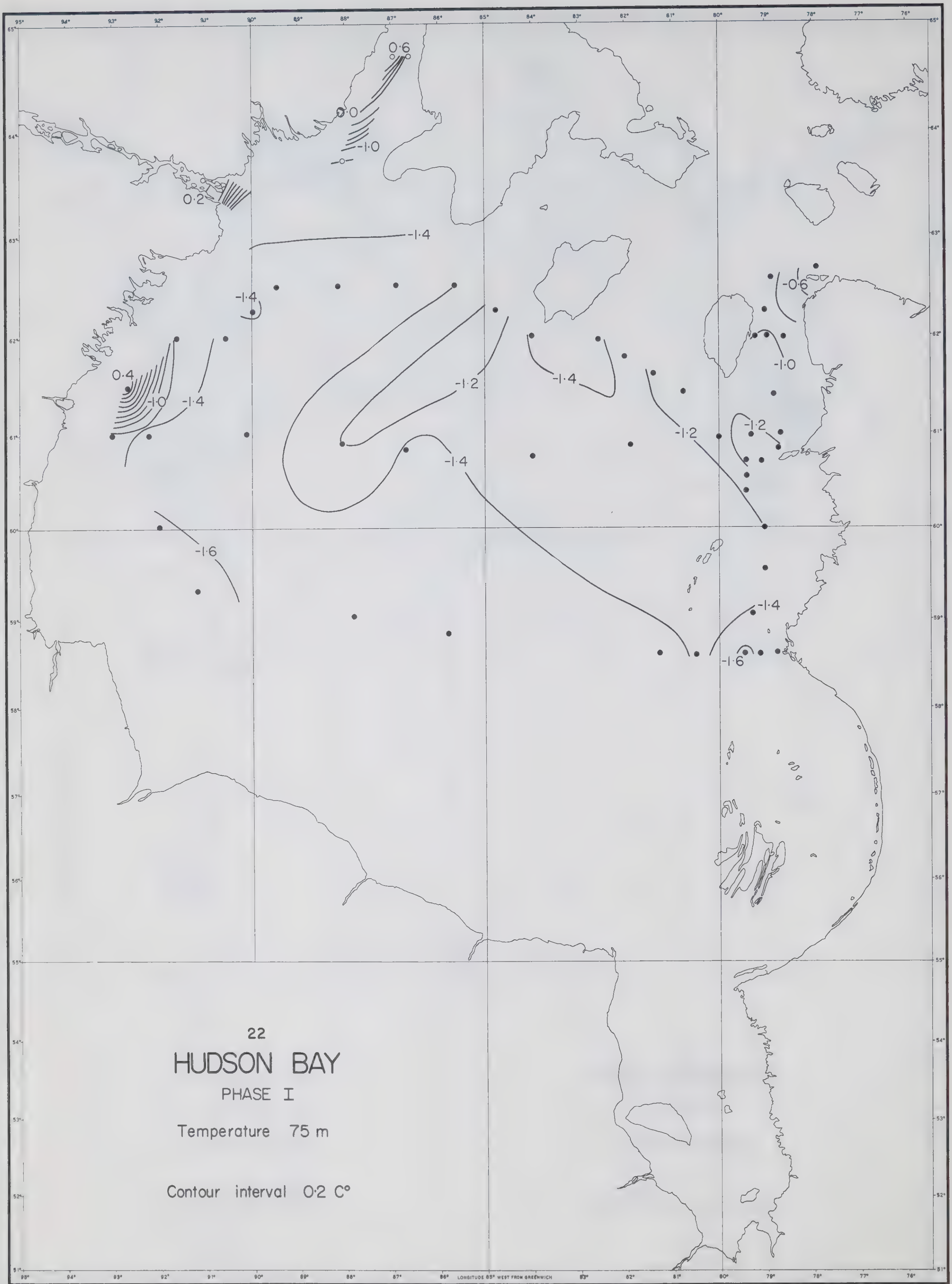
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176

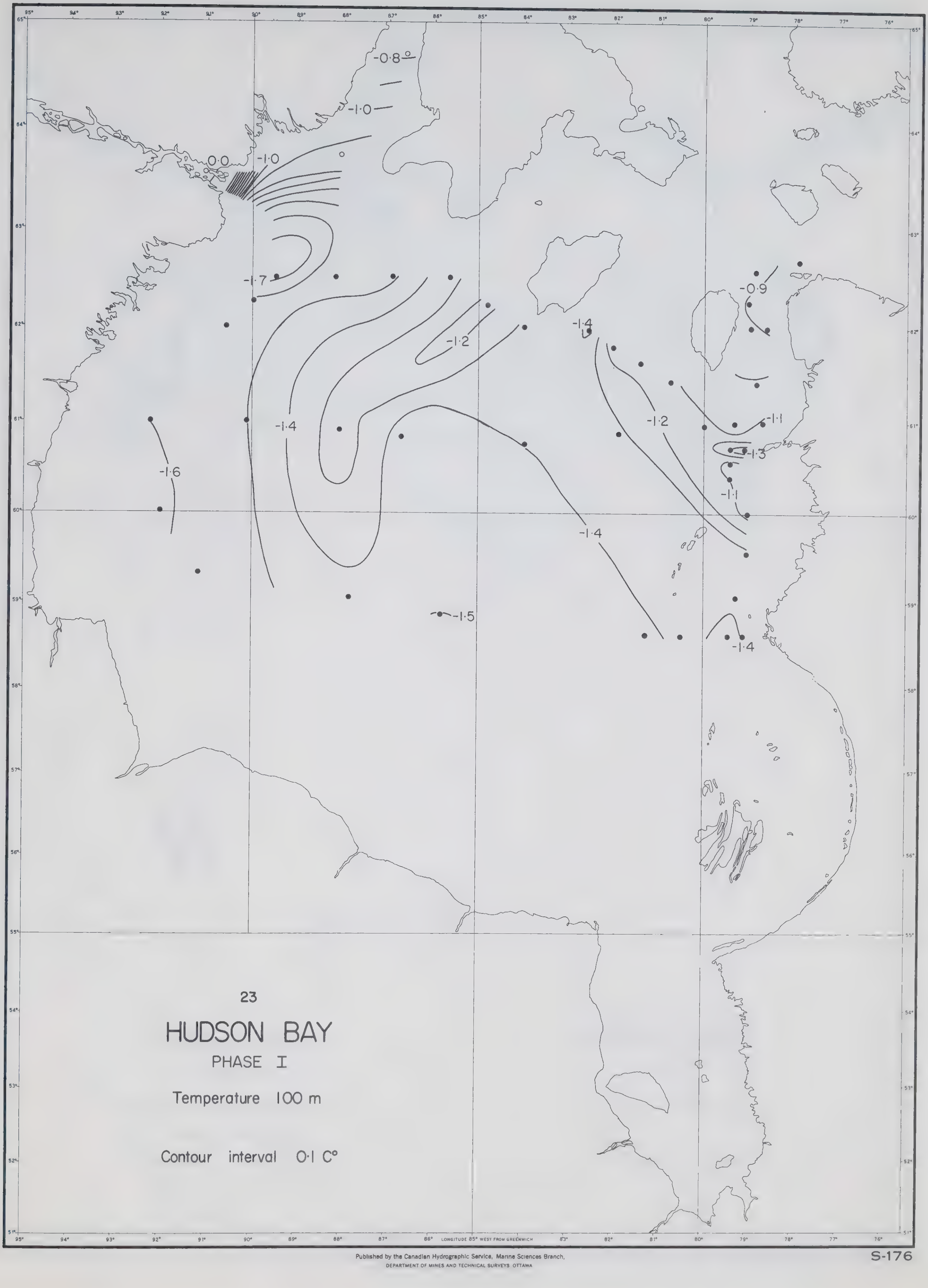


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA

S-176



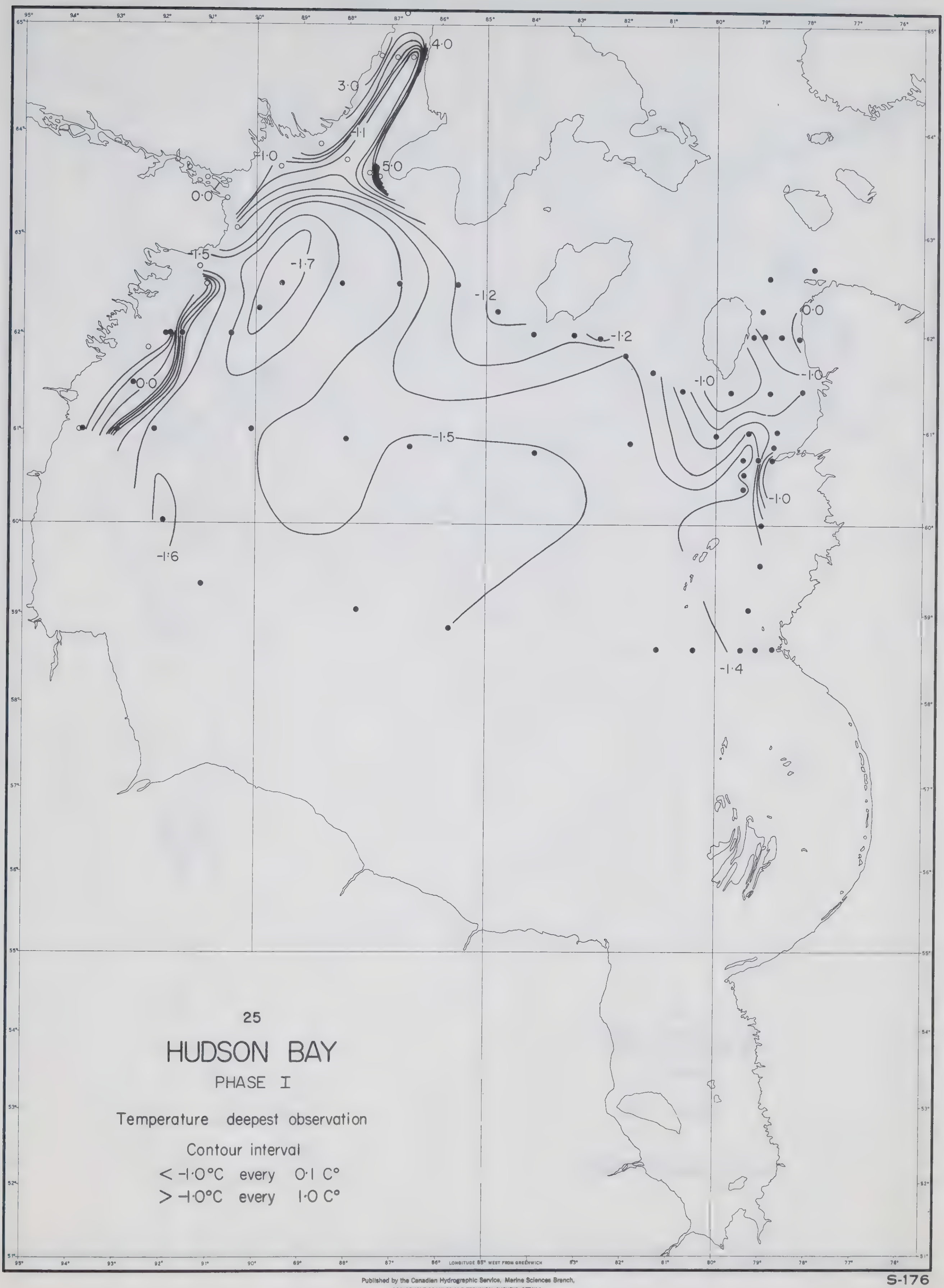
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



25

HUDSON BAY

PHASE I

Temperature deepest observation

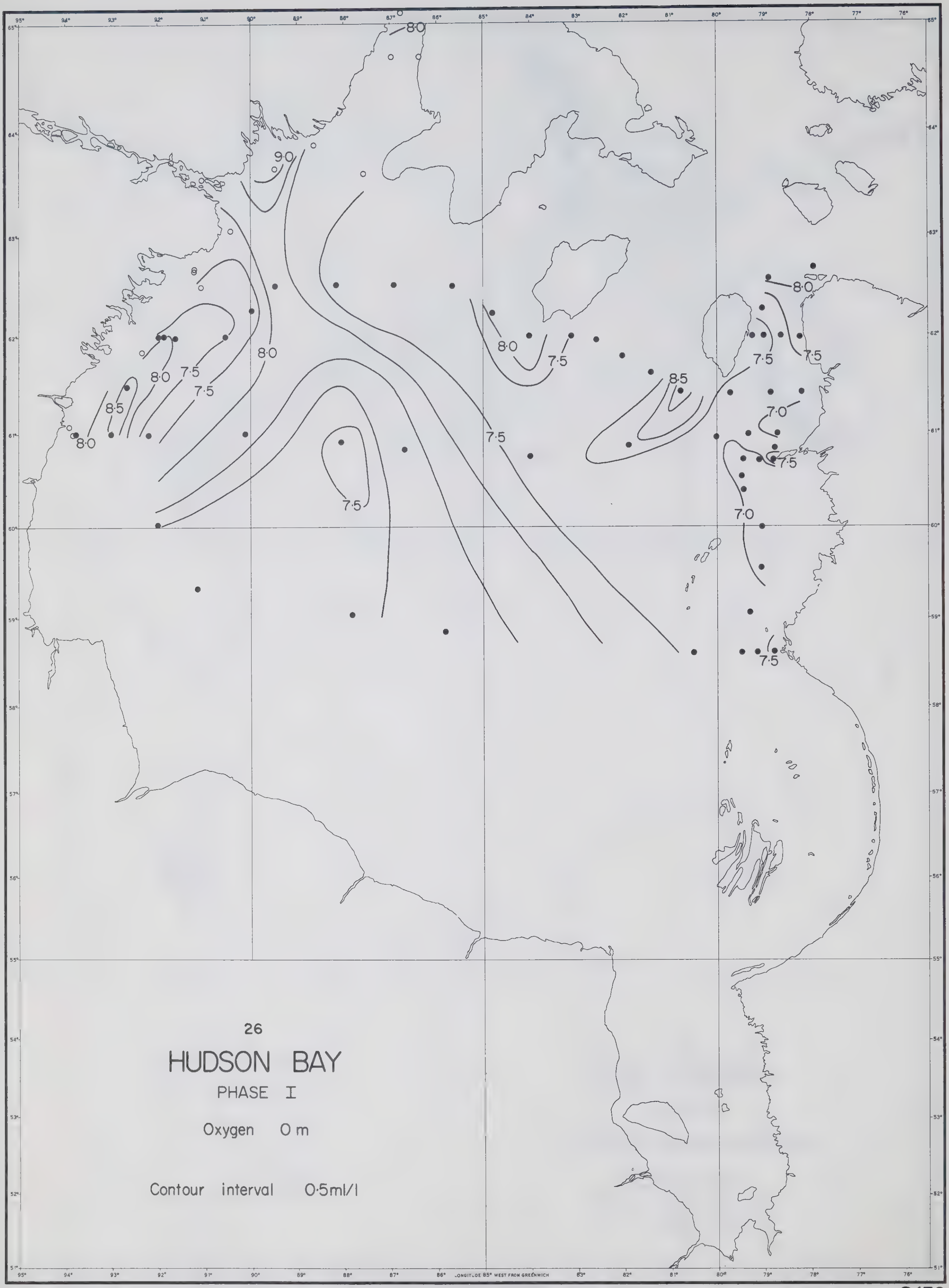
Contour interval

$< -1{\cdot}0$°C every 0·1 C°

$> -1{\cdot}0$°C every 1·0 C°

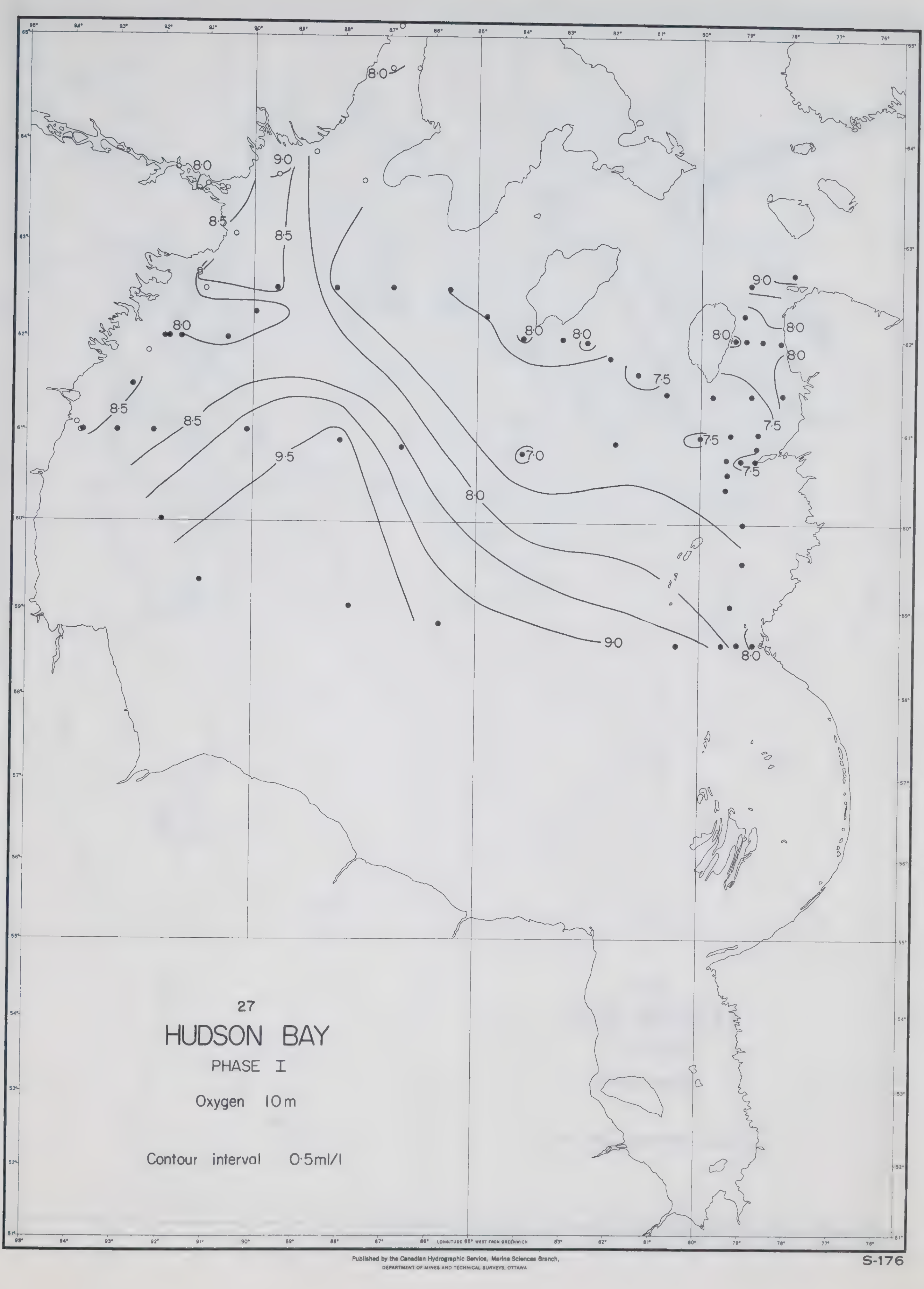
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



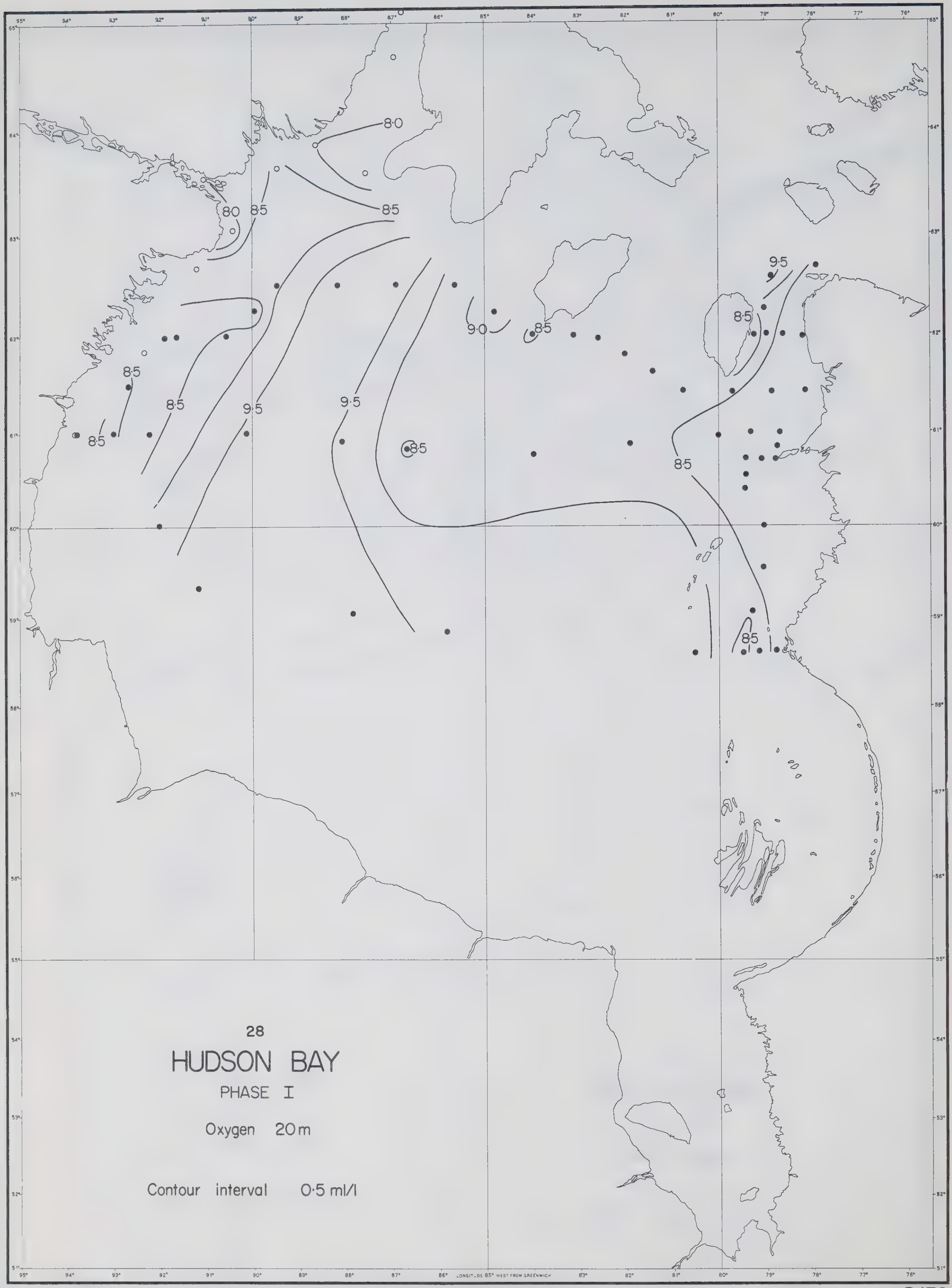
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



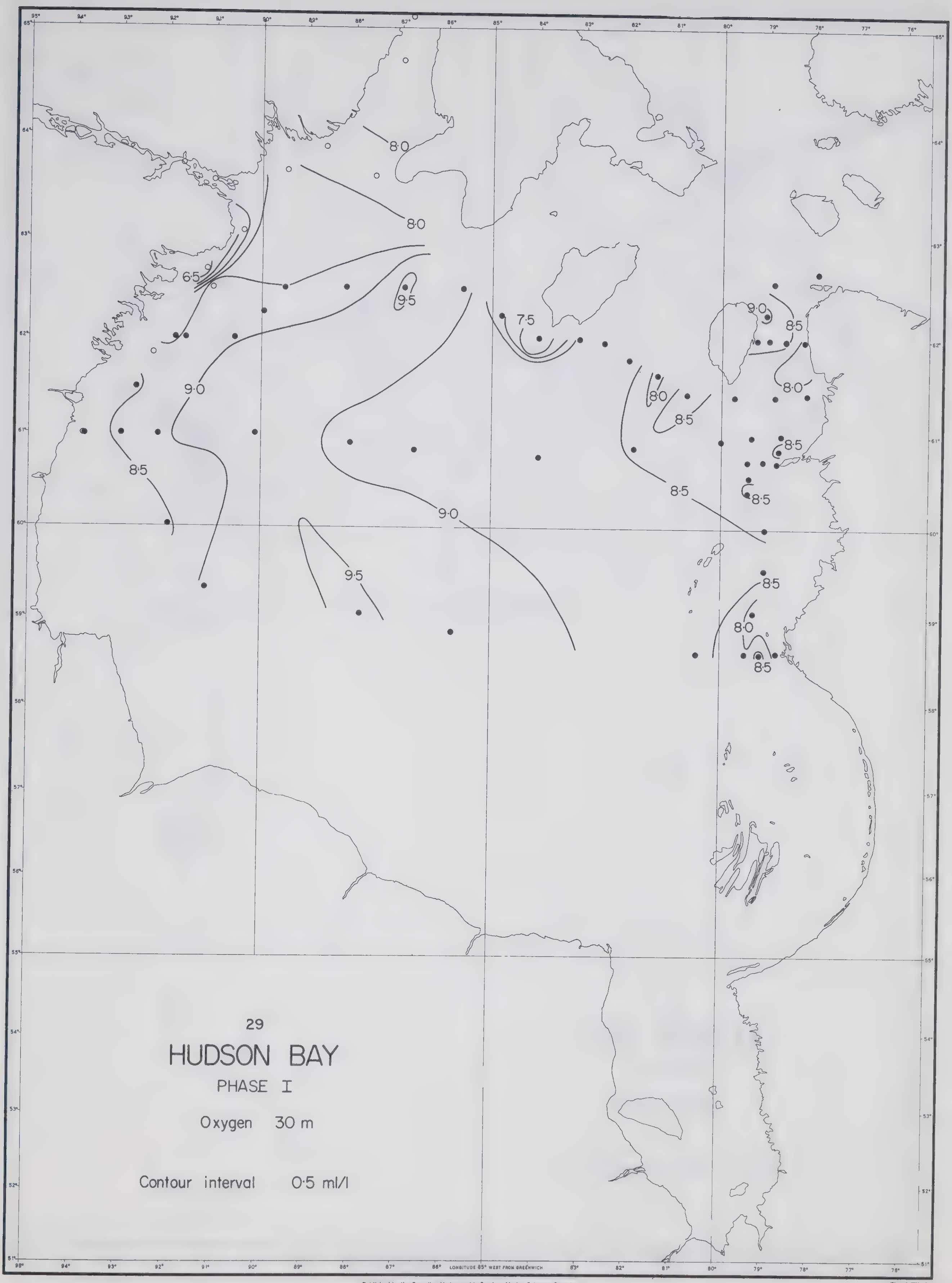
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



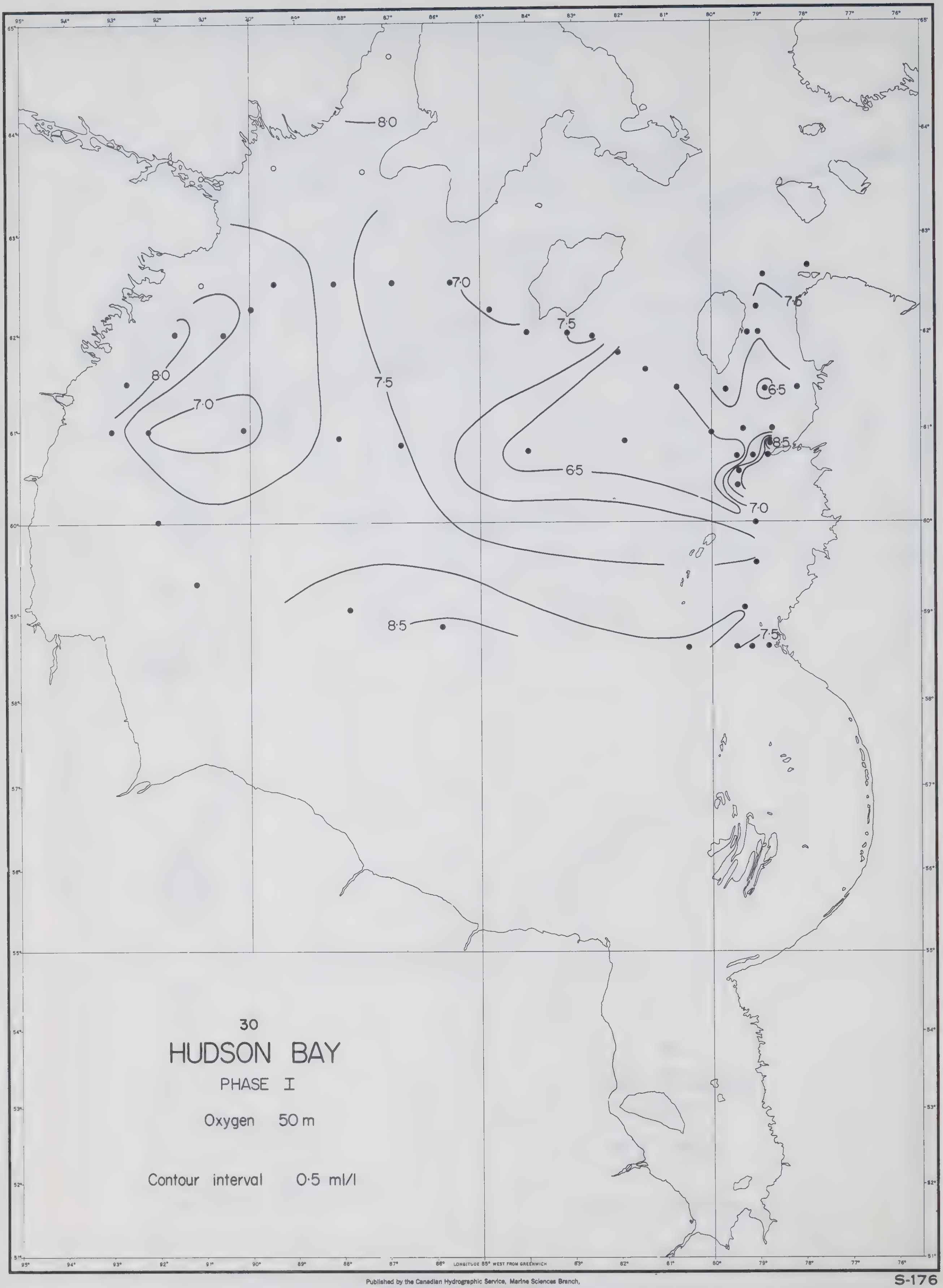
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



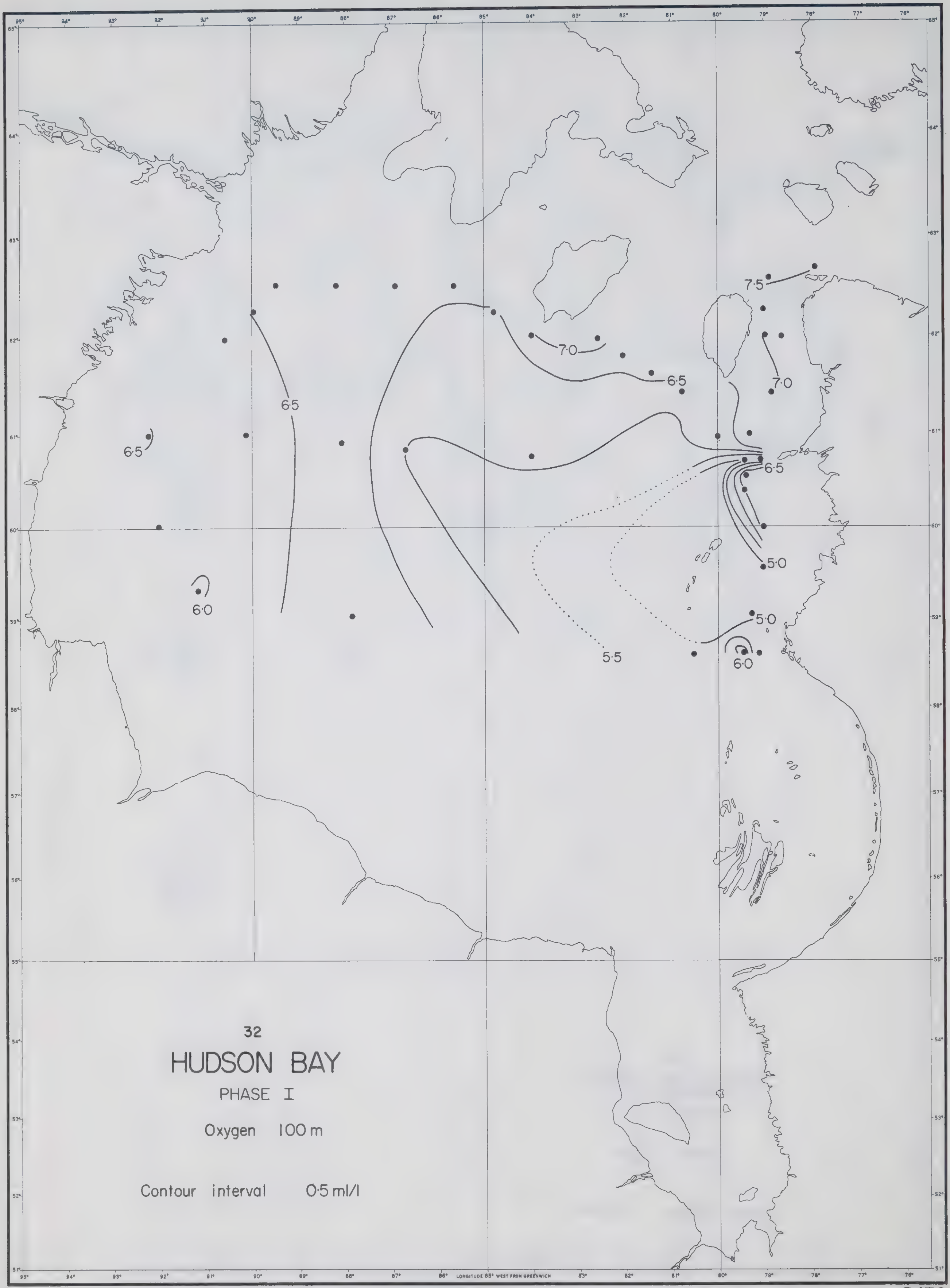
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



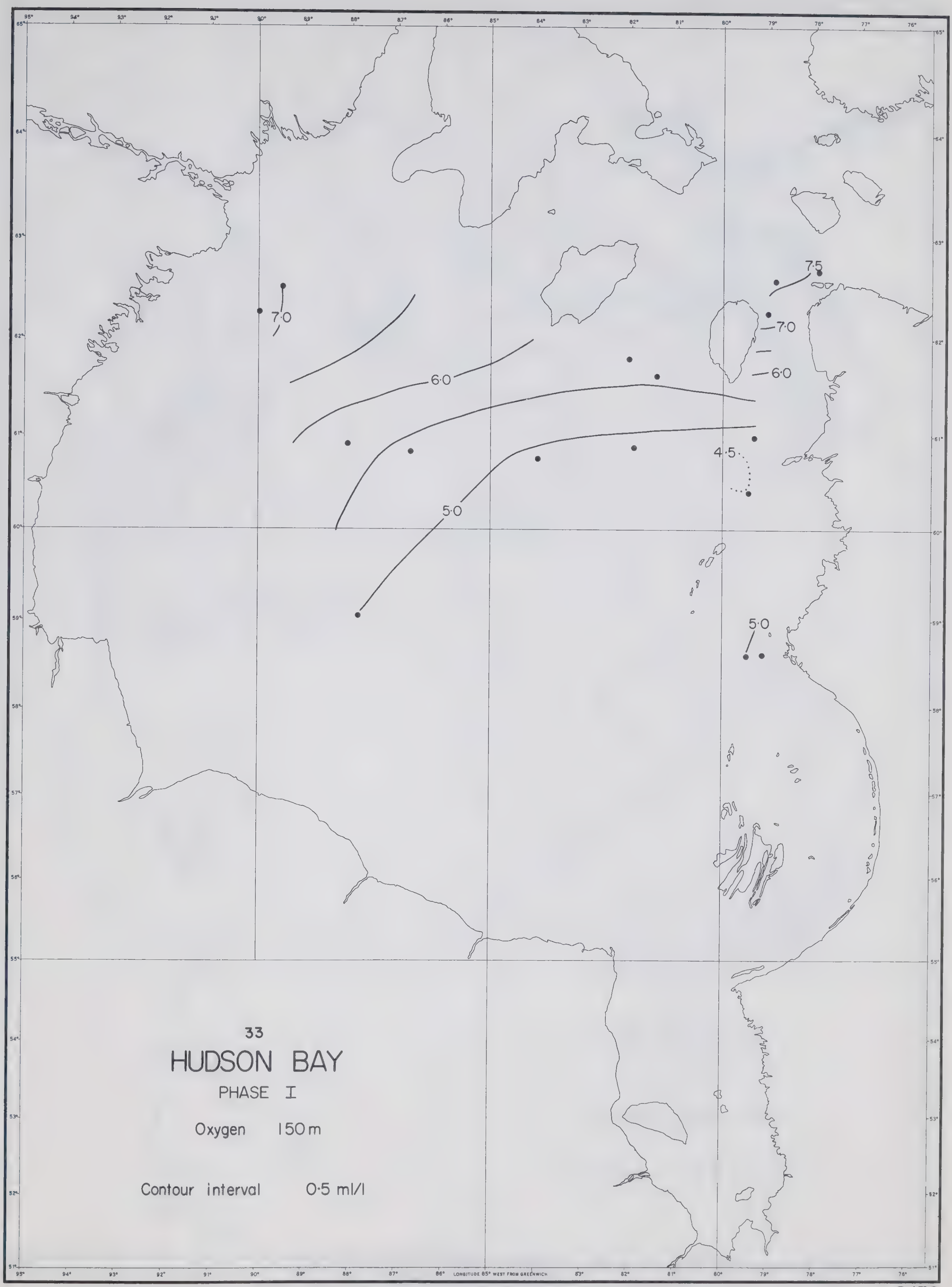
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



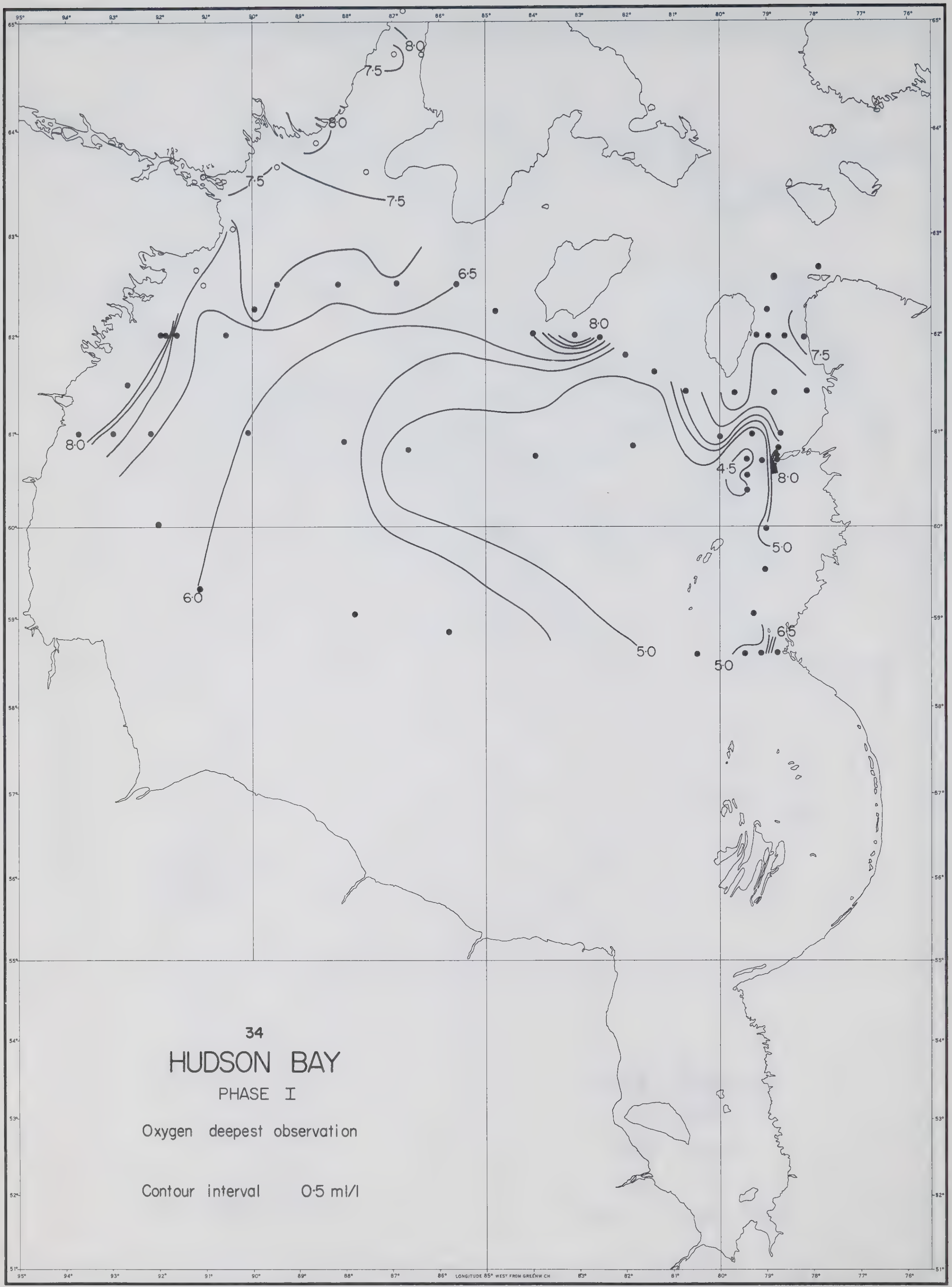
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



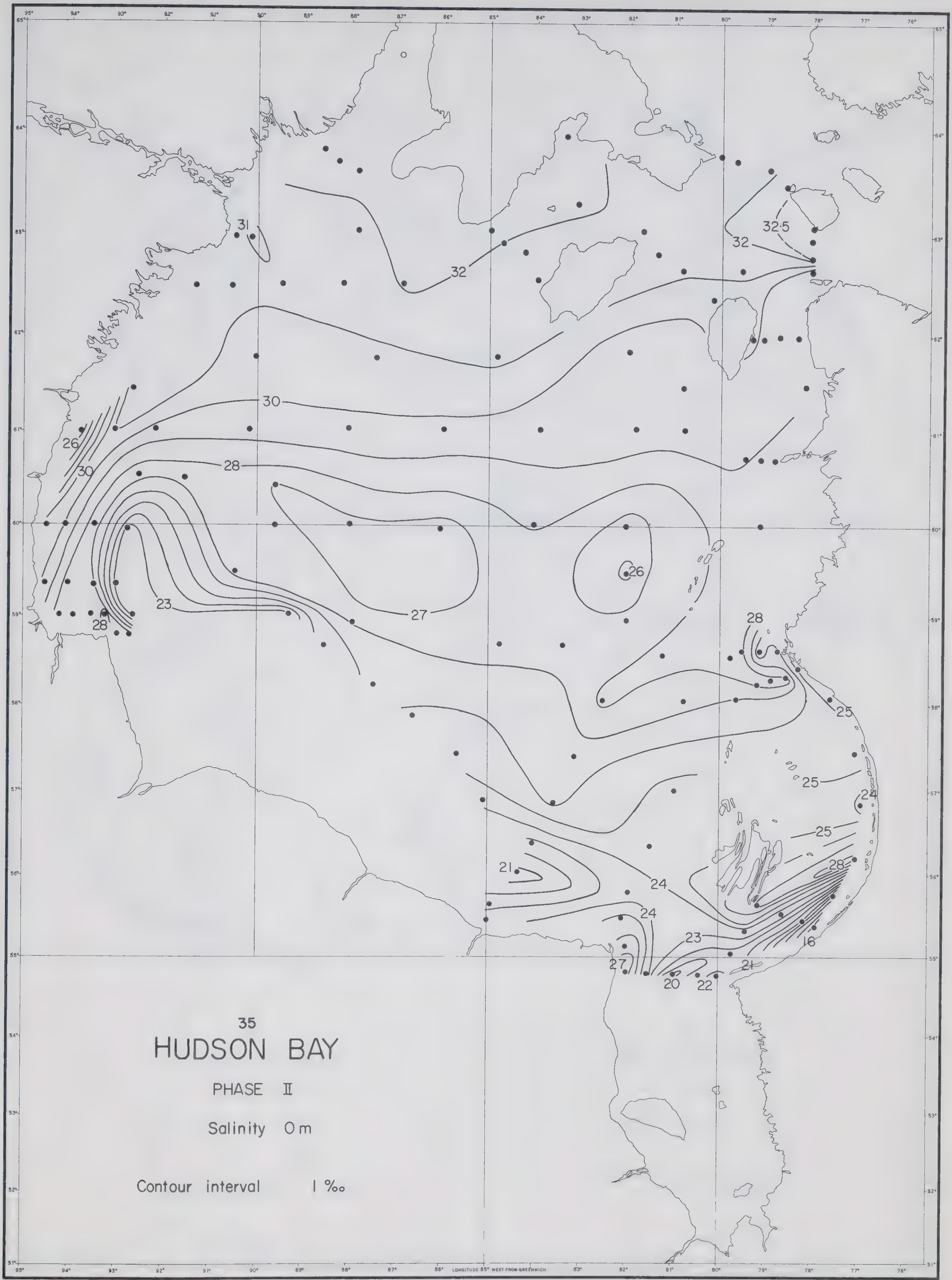
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



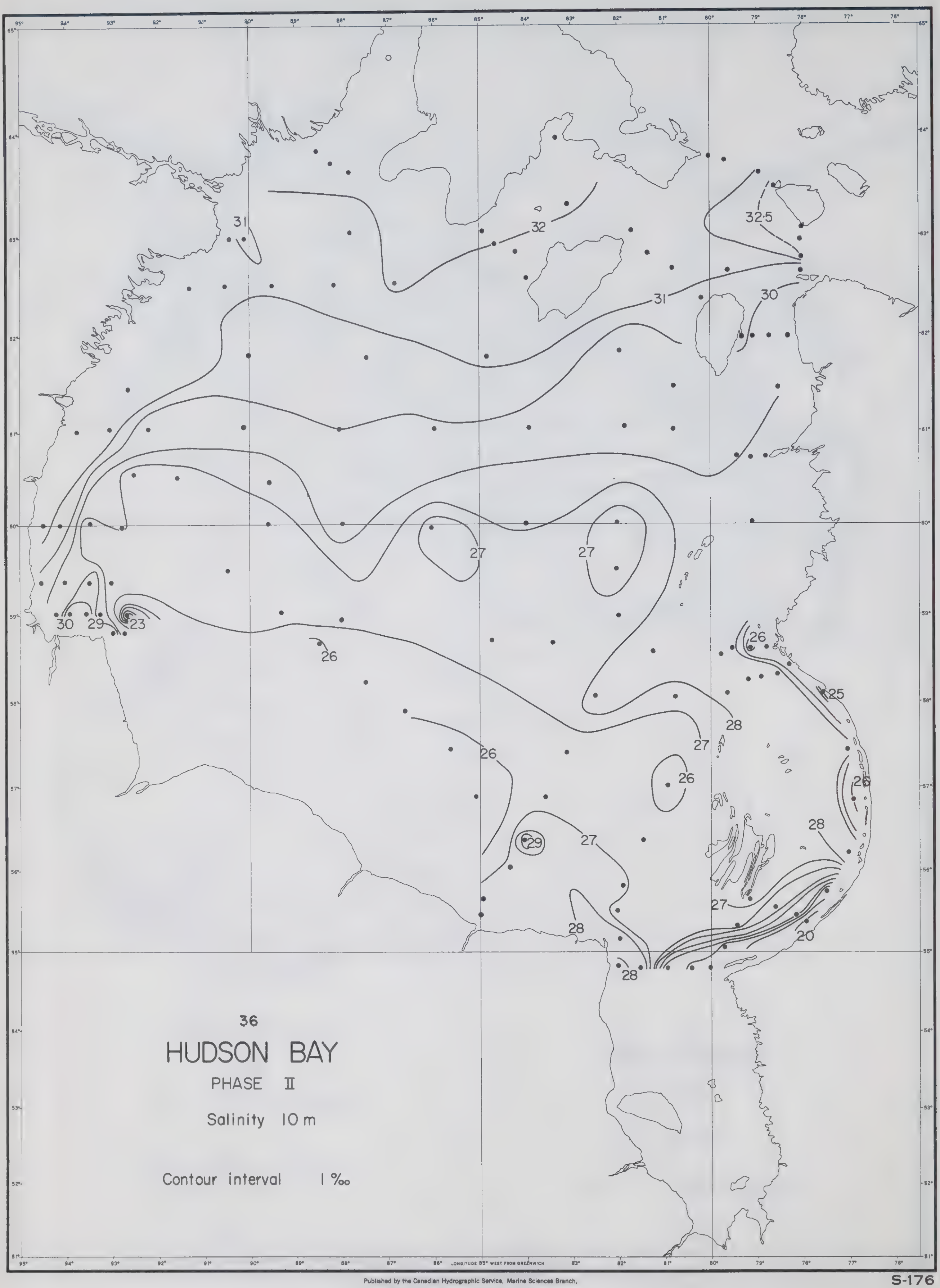
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



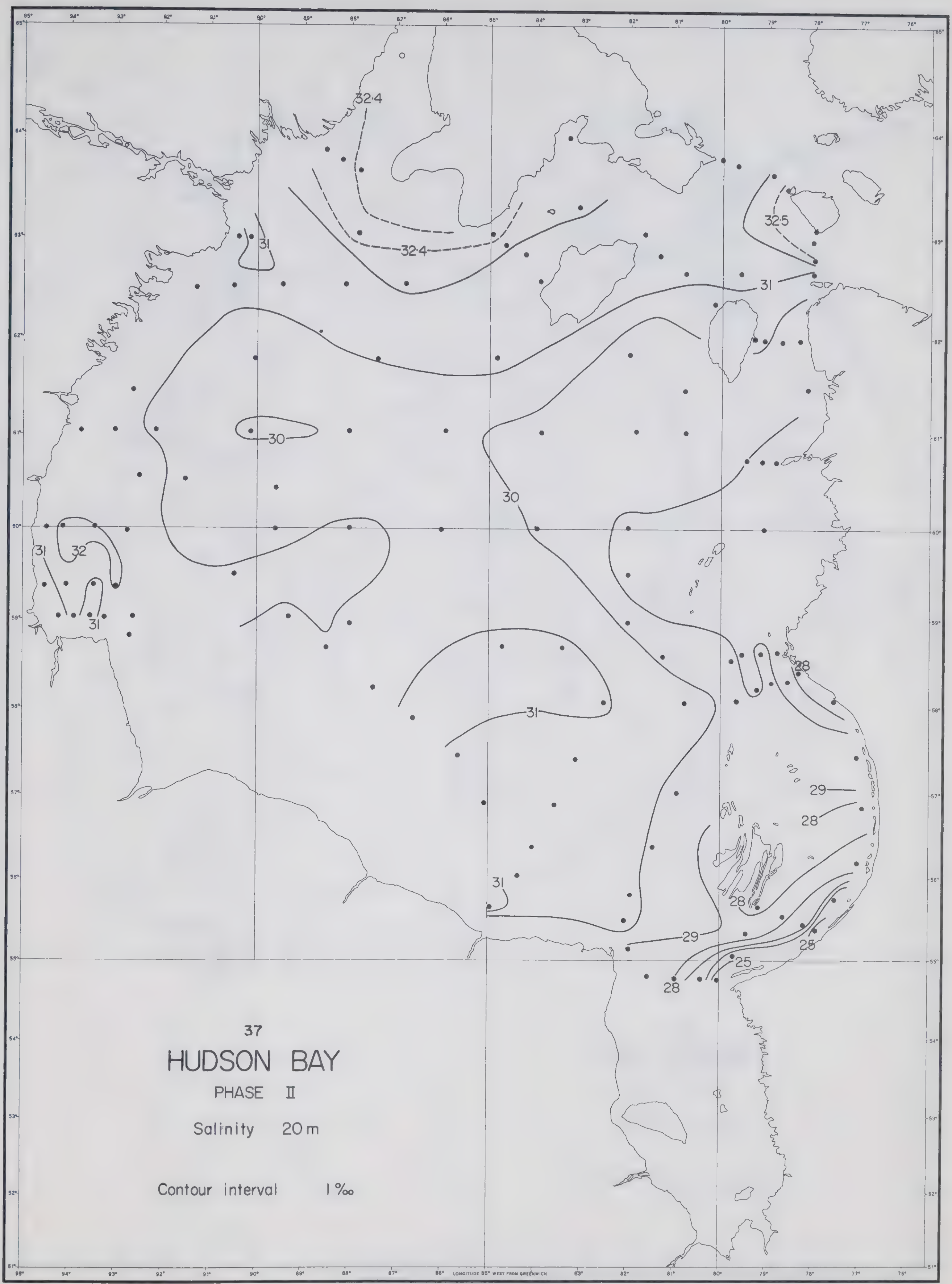
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



37

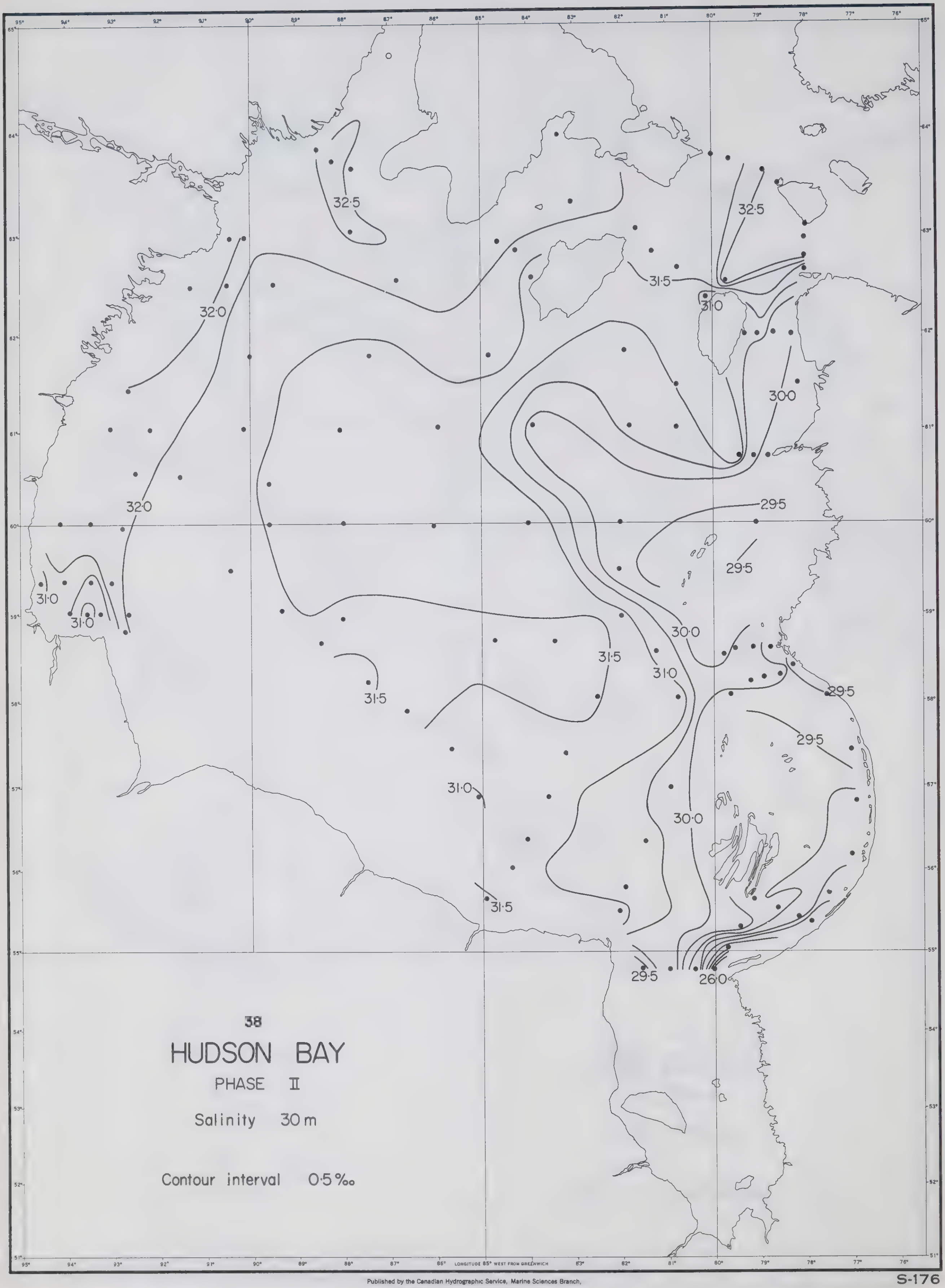
HUDSON BAY

PHASE II

Salinity 20 m

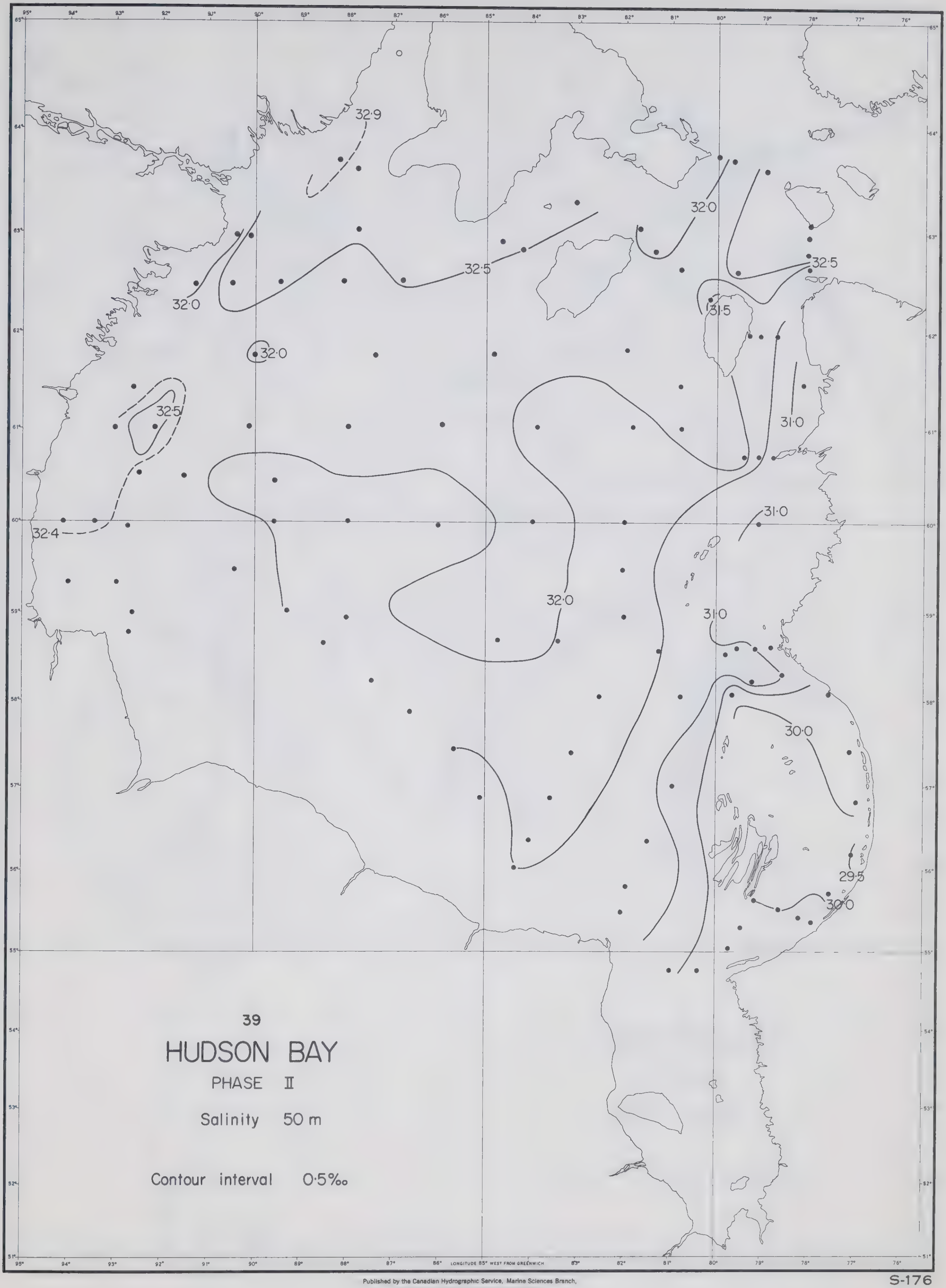
Contour interval 1 ‰

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



39

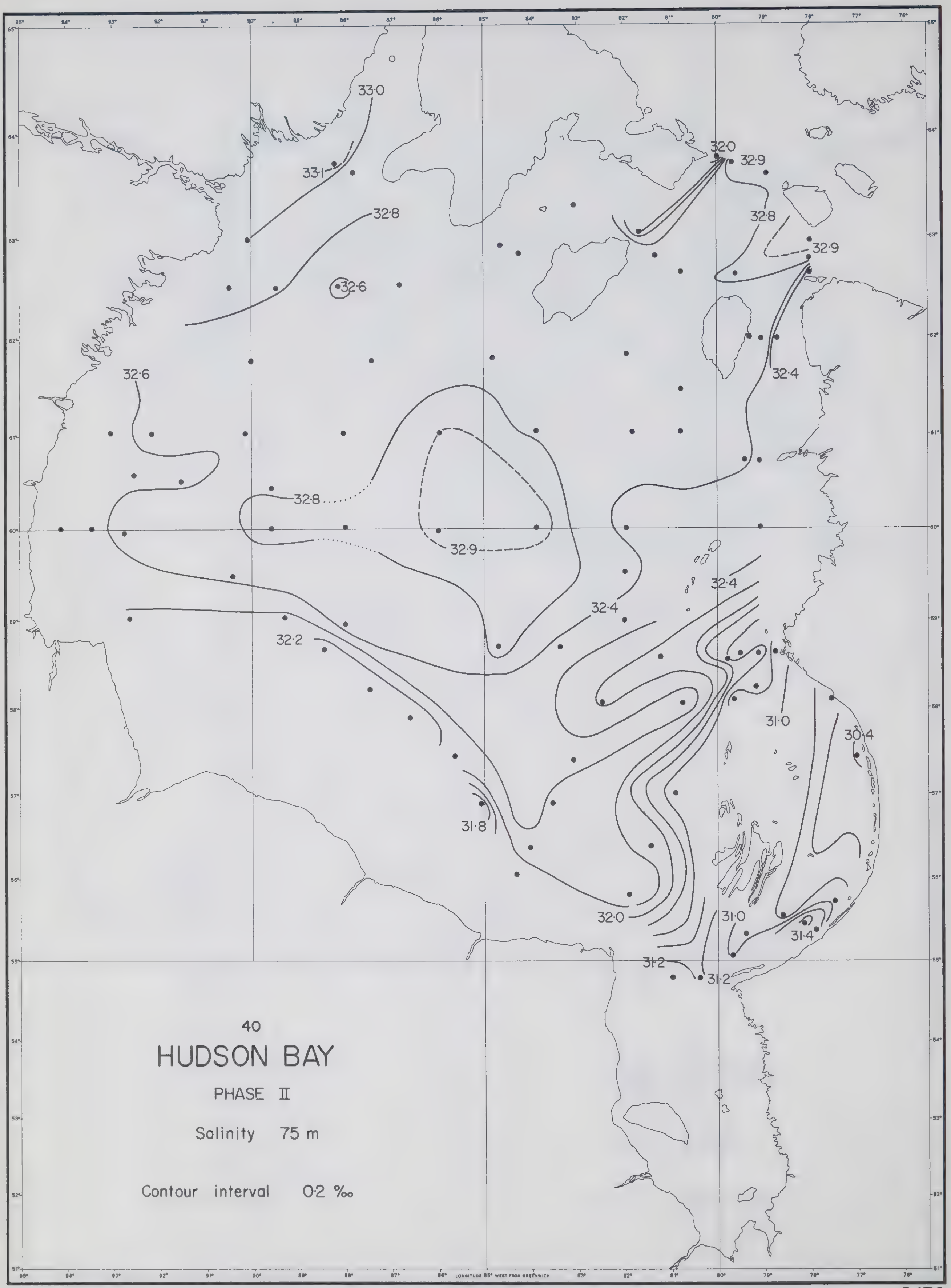
HUDSON BAY

PHASE II

Salinity 50 m

Contour interval 0·5‰

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



40

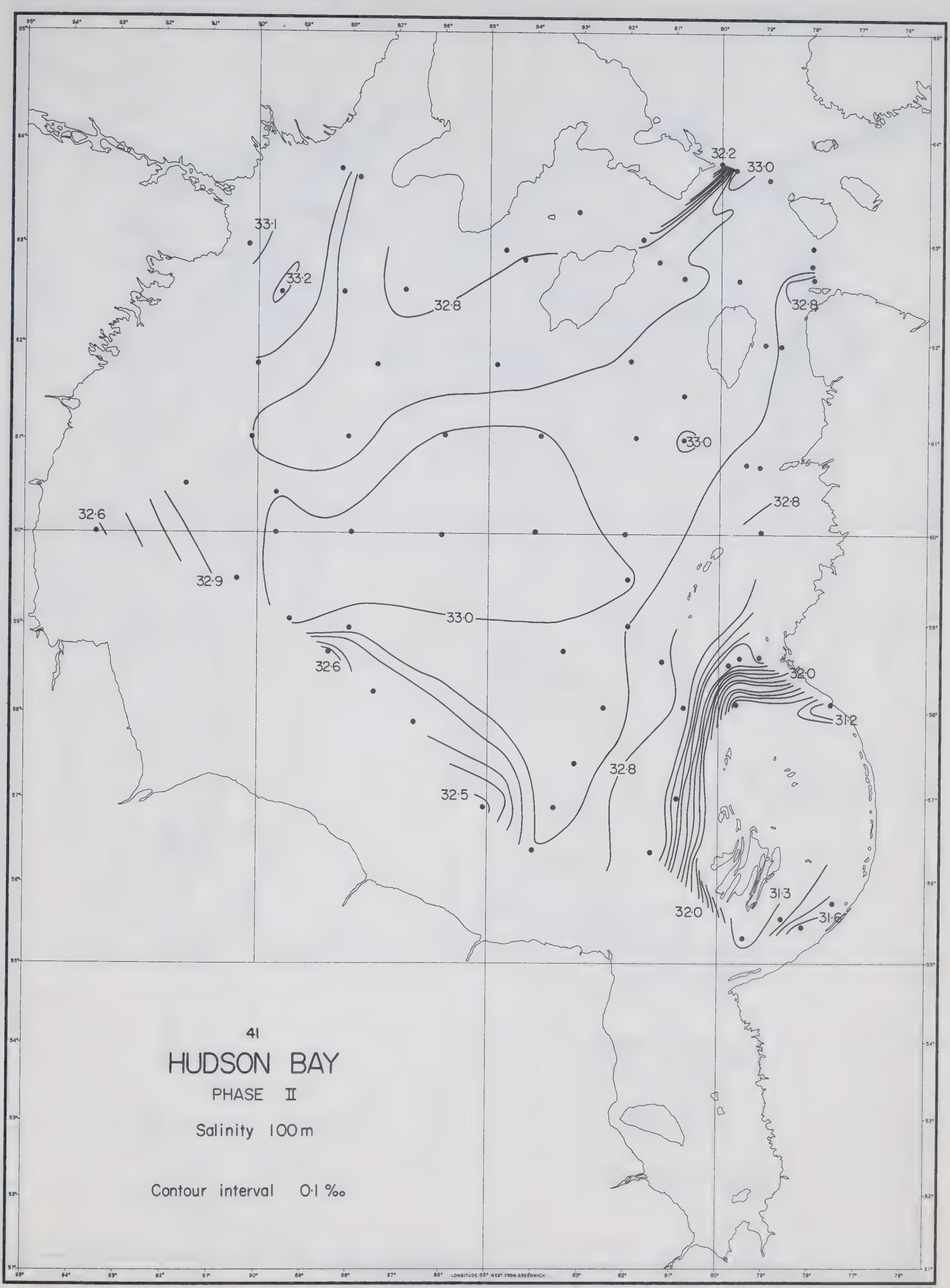
HUDSON BAY

PHASE II

Salinity 75 m

Contour interval 0·2 ‰

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



41

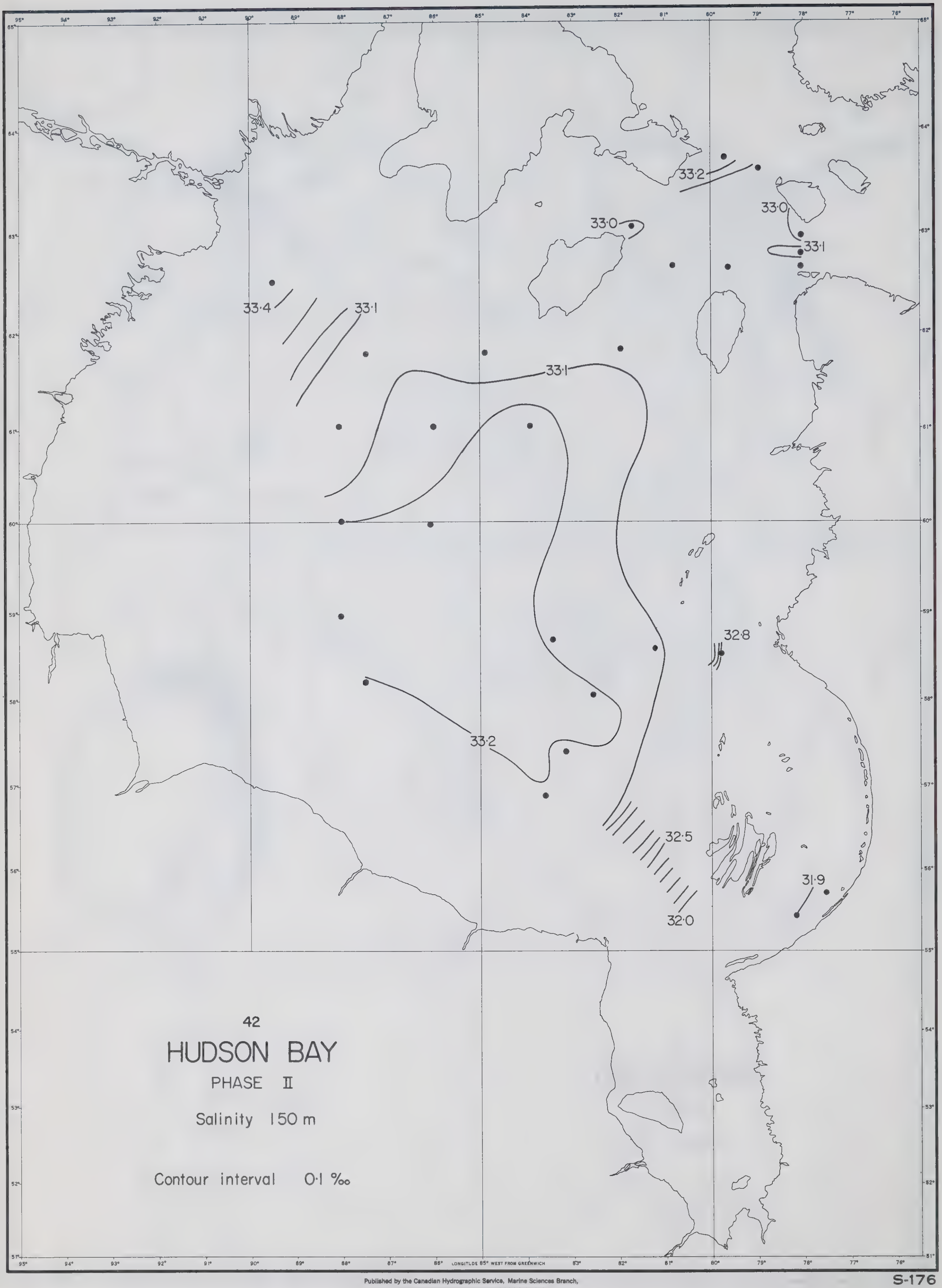
HUDSON BAY

PHASE II

Salinity 100 m

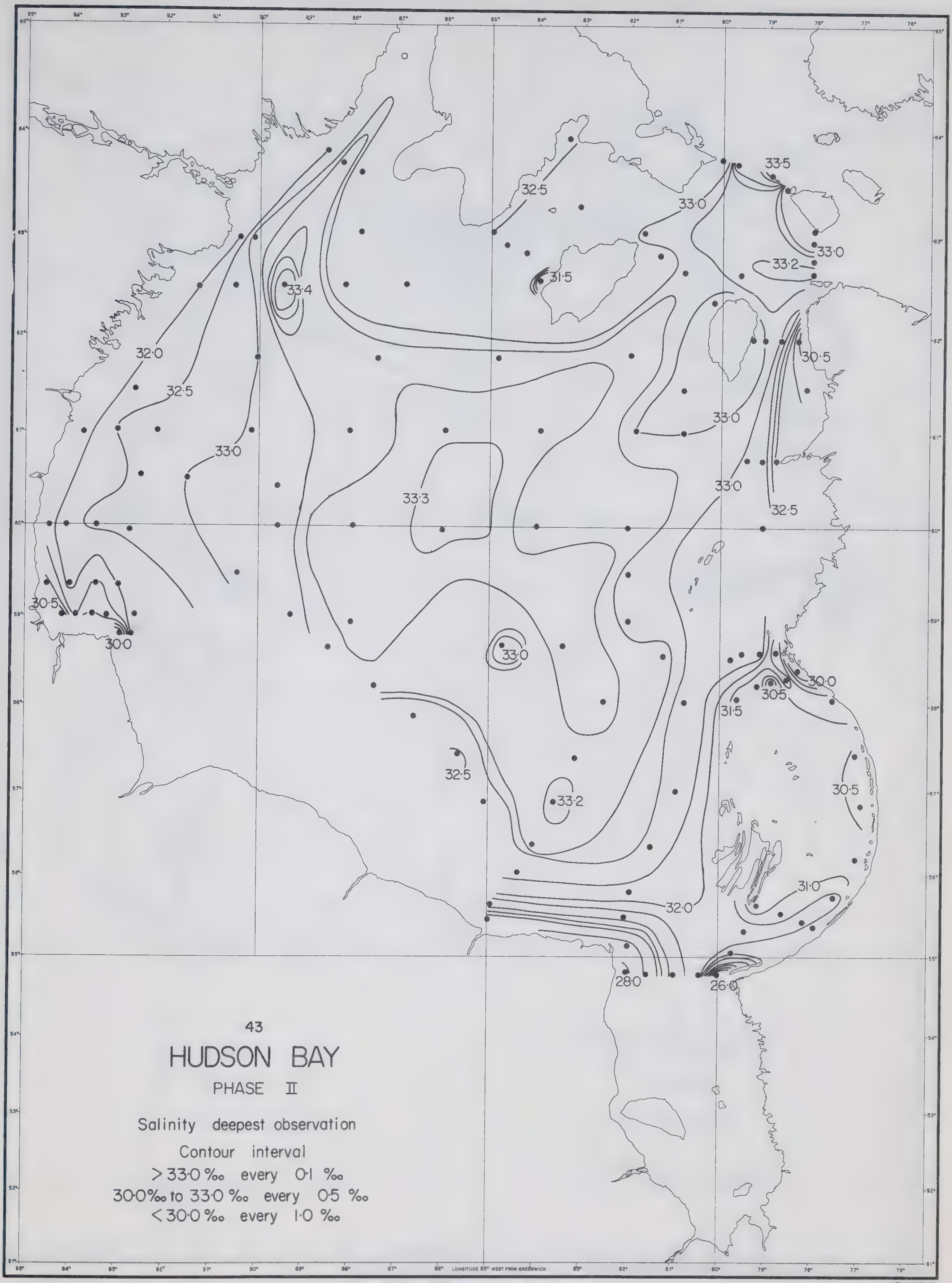
Contour interval 0·1 ‰

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

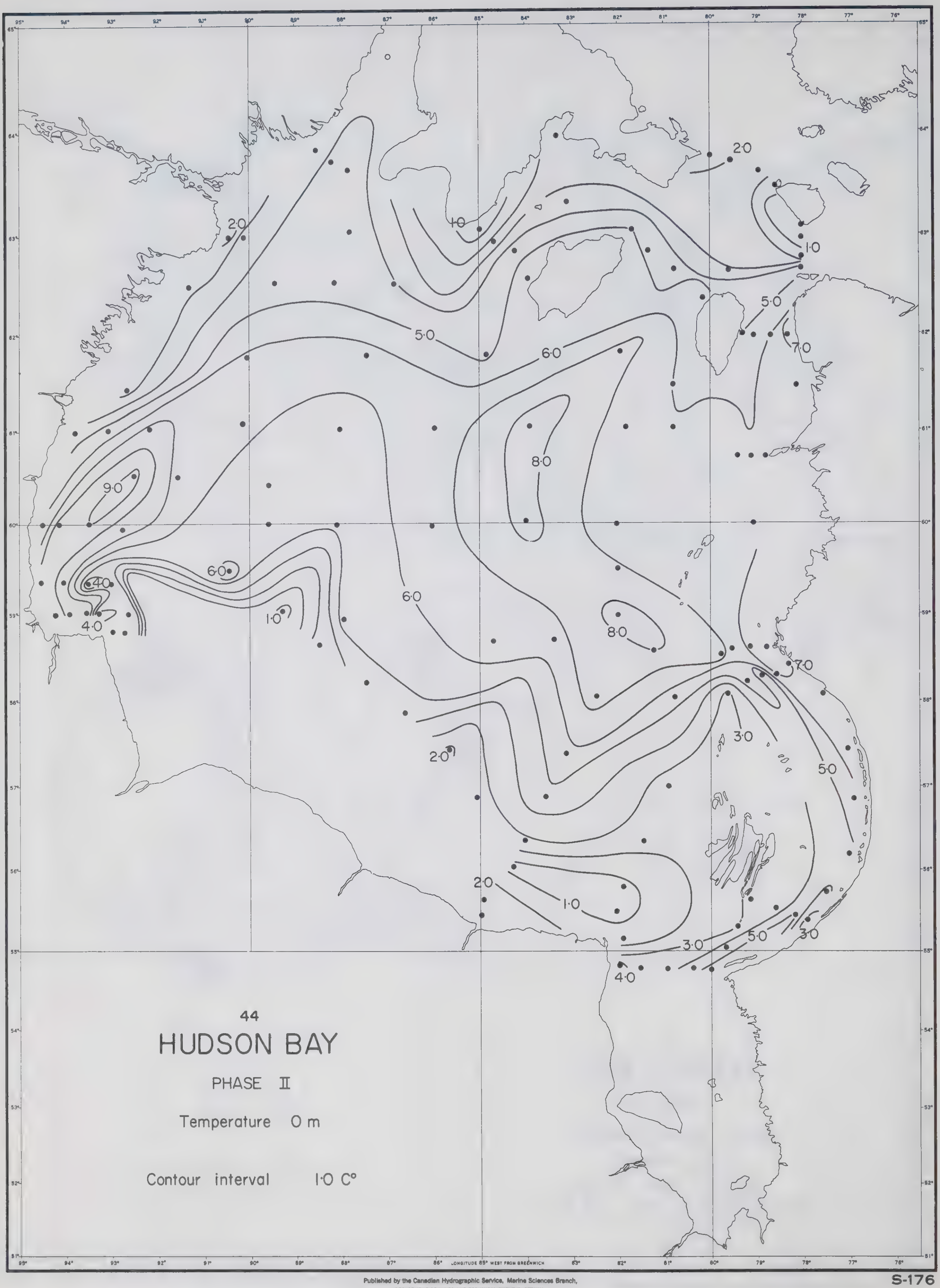


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



44

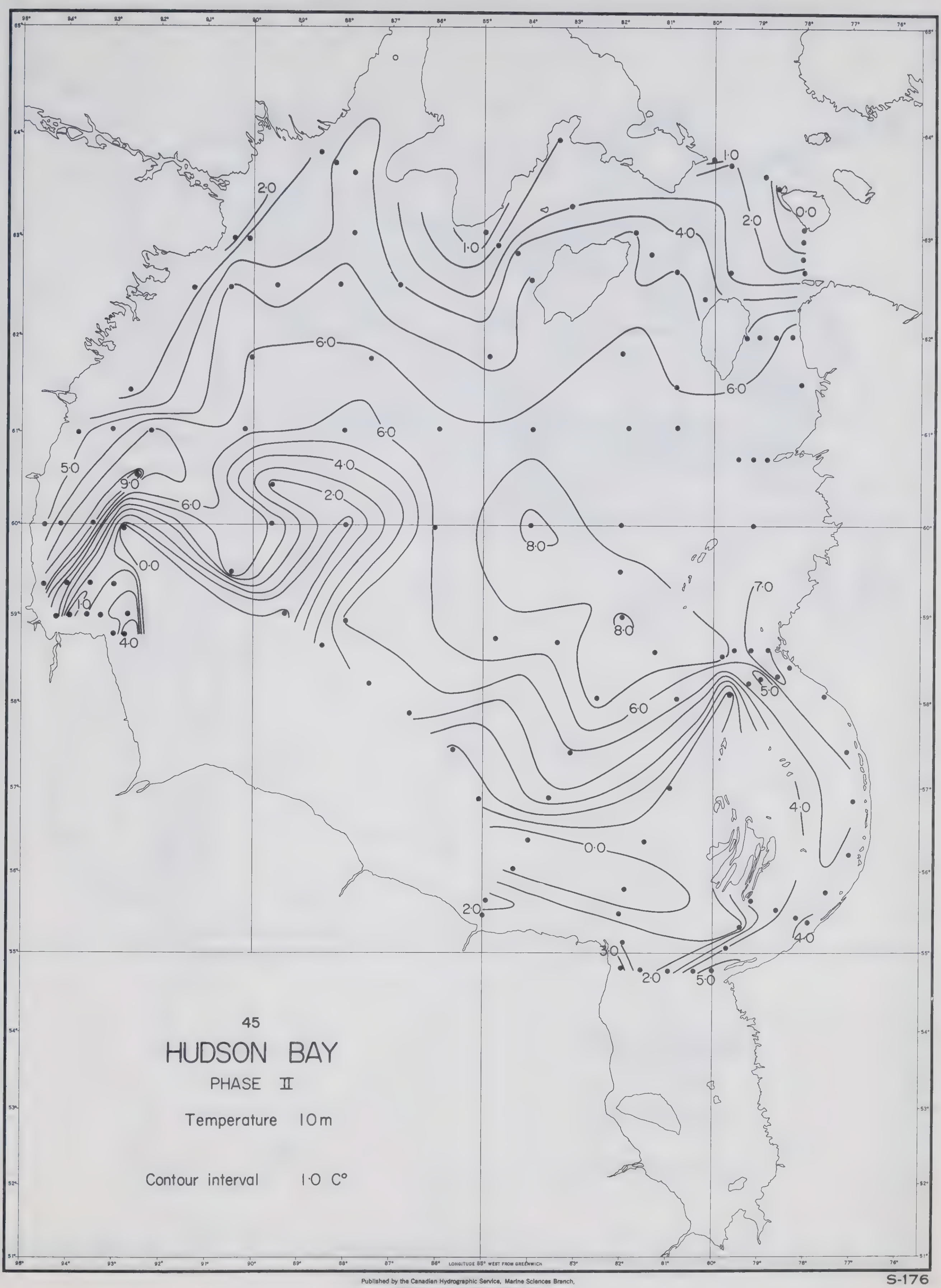
HUDSON BAY

PHASE II

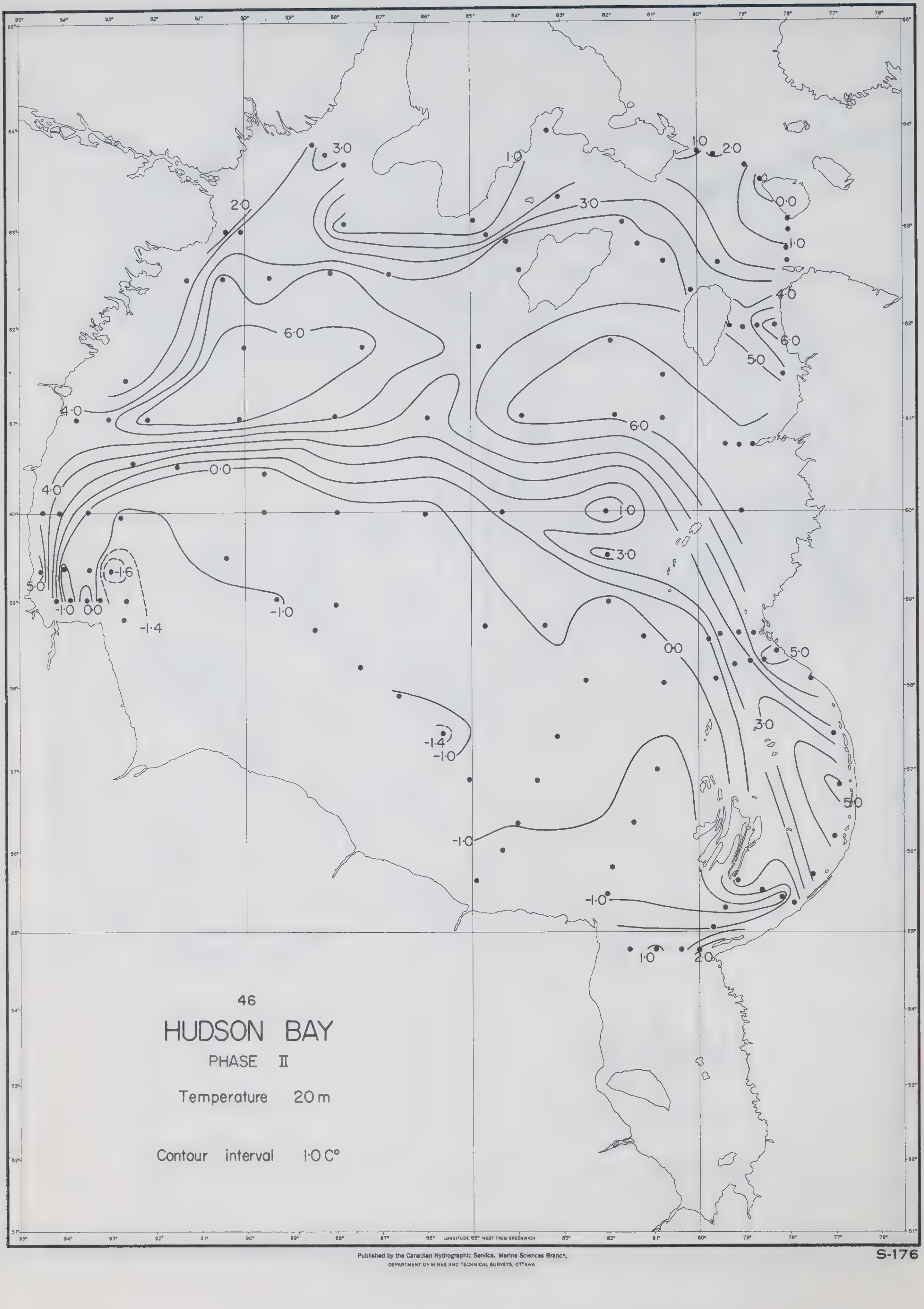
Temperature 0 m

Contour interval 1·0 C°

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



46

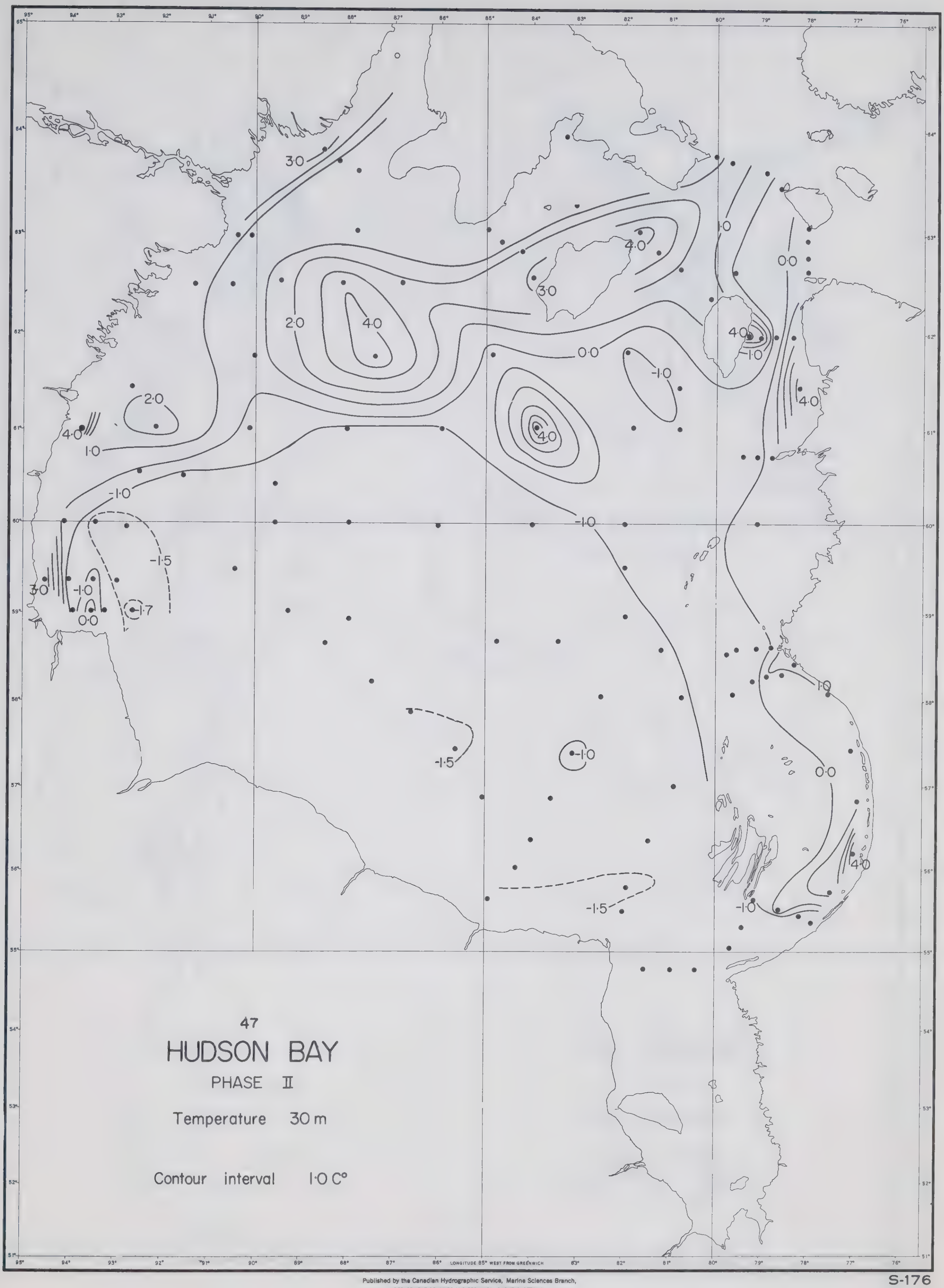
HUDSON BAY

PHASE II

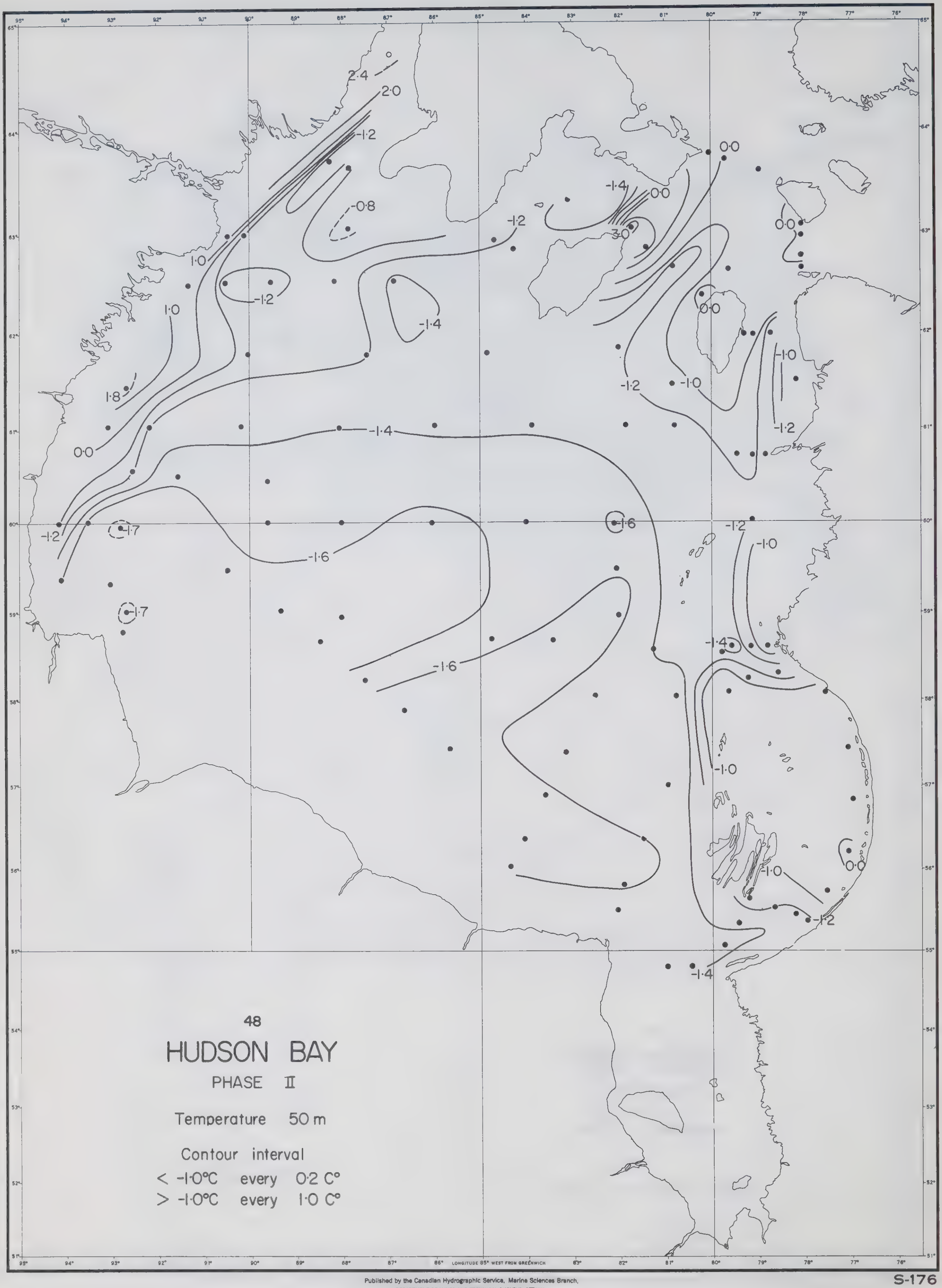
Temperature 20 m

Contour interval 1·0 C°

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

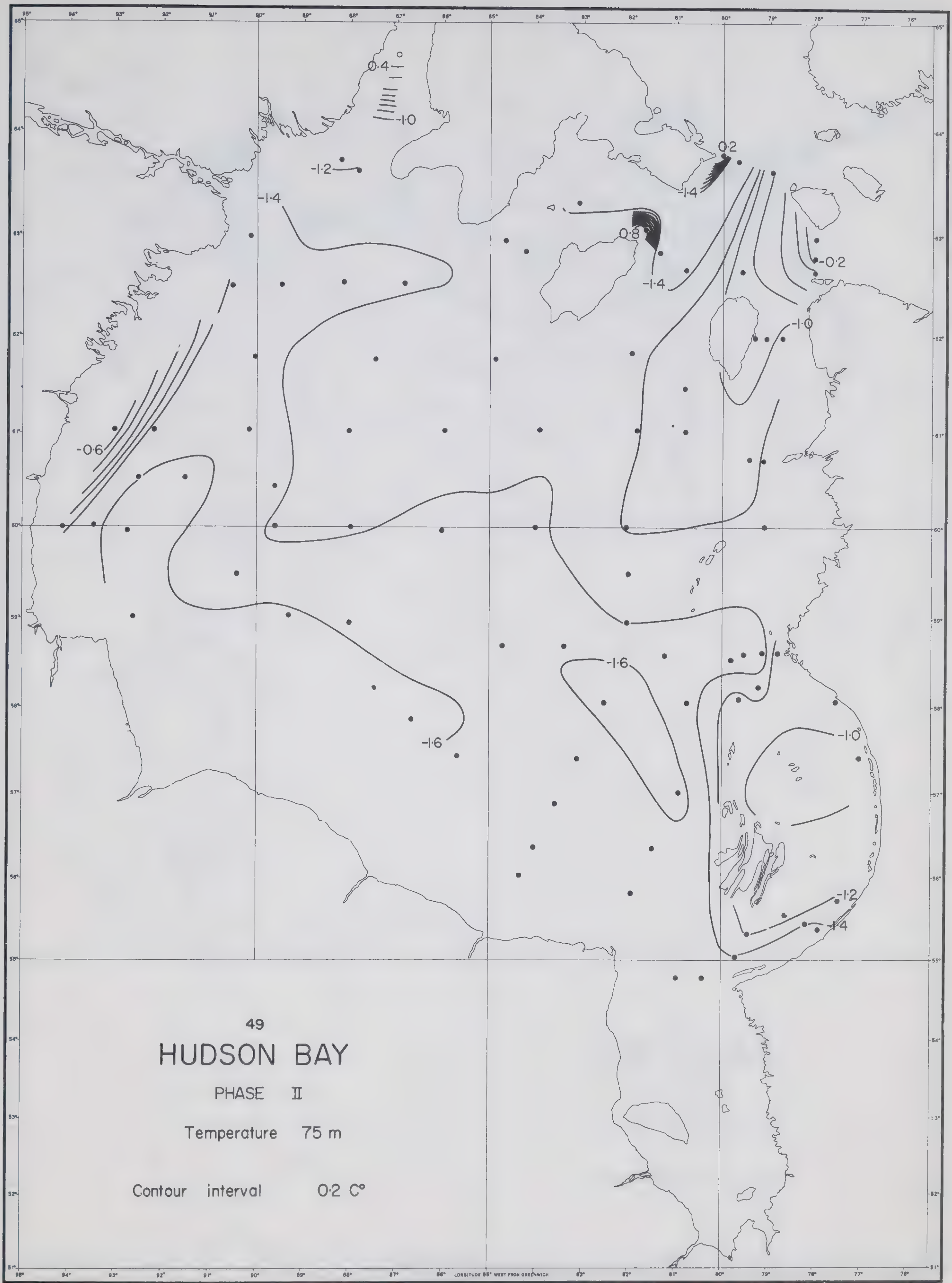


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



49

HUDSON BAY

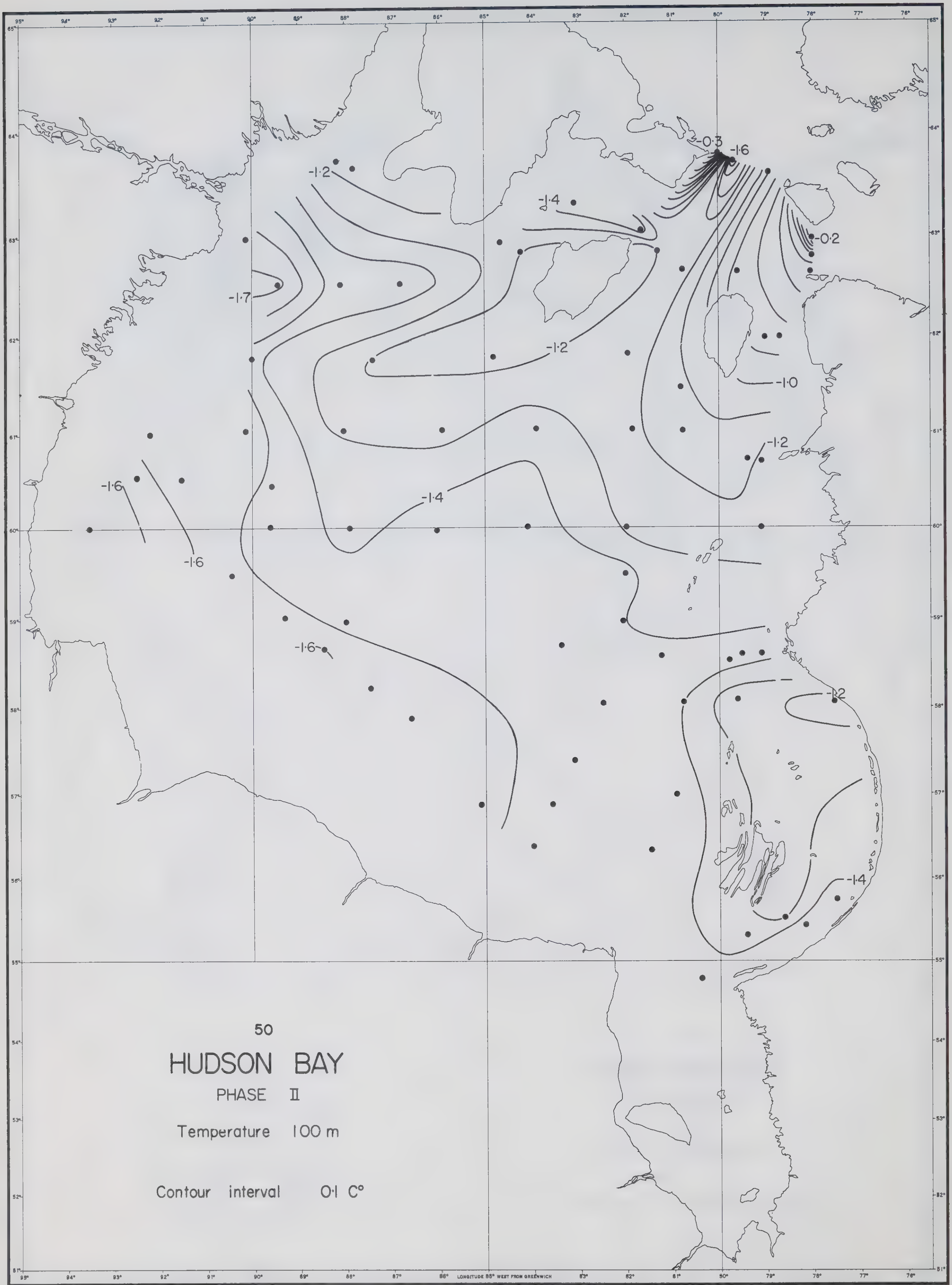
PHASE II

Temperature 75 m

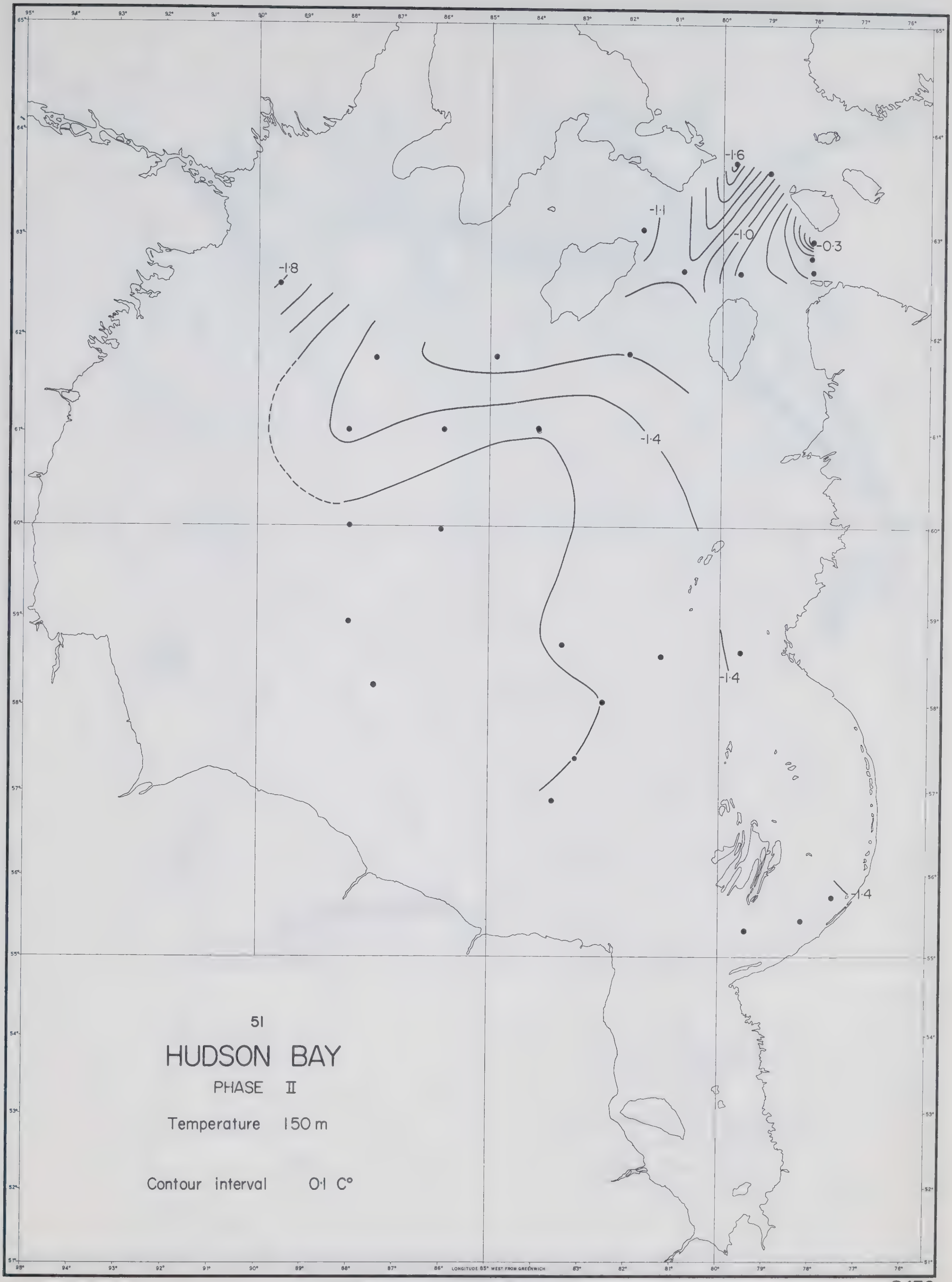
Contour interval 0·2 C°

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

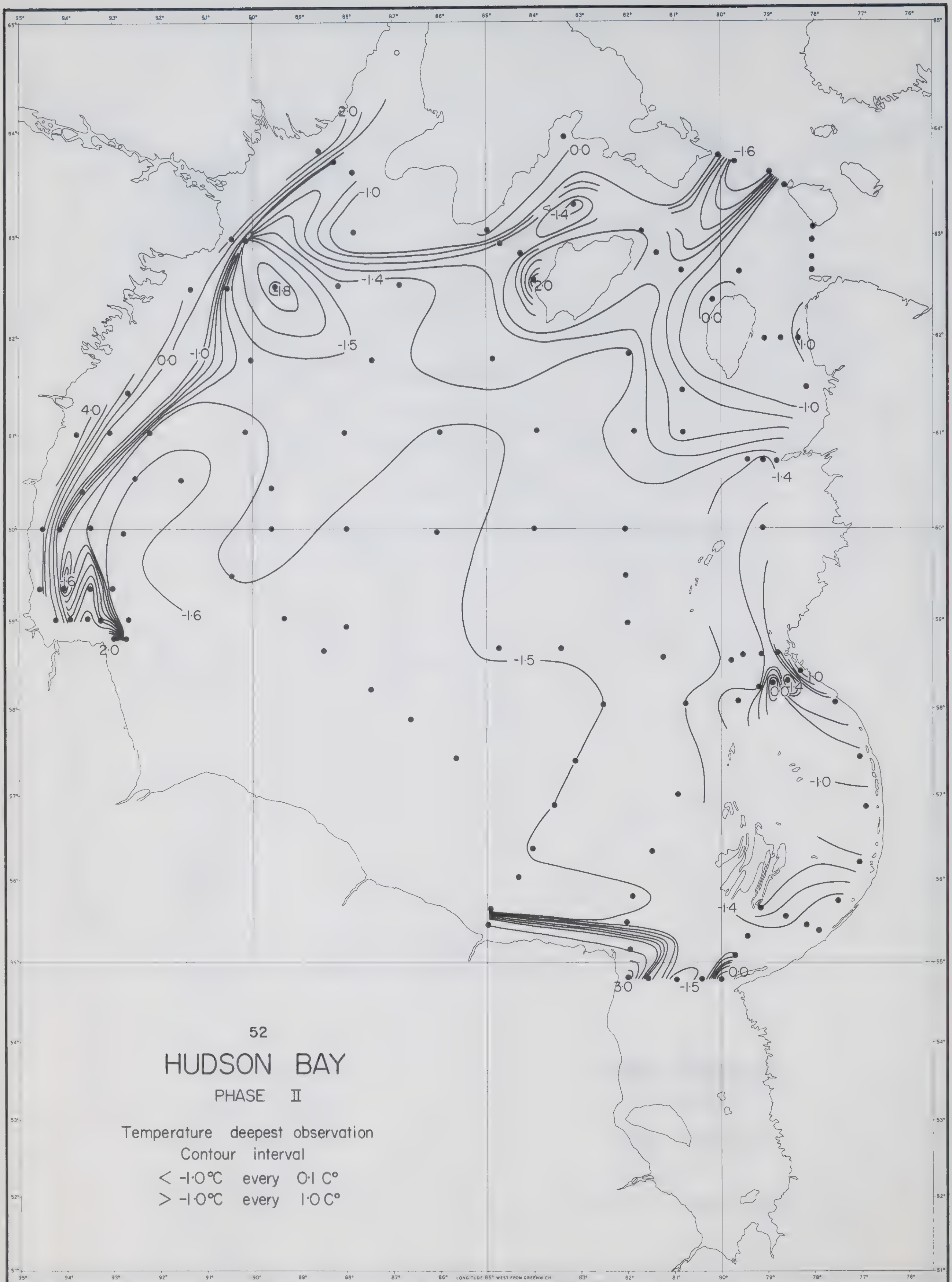
S-176



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



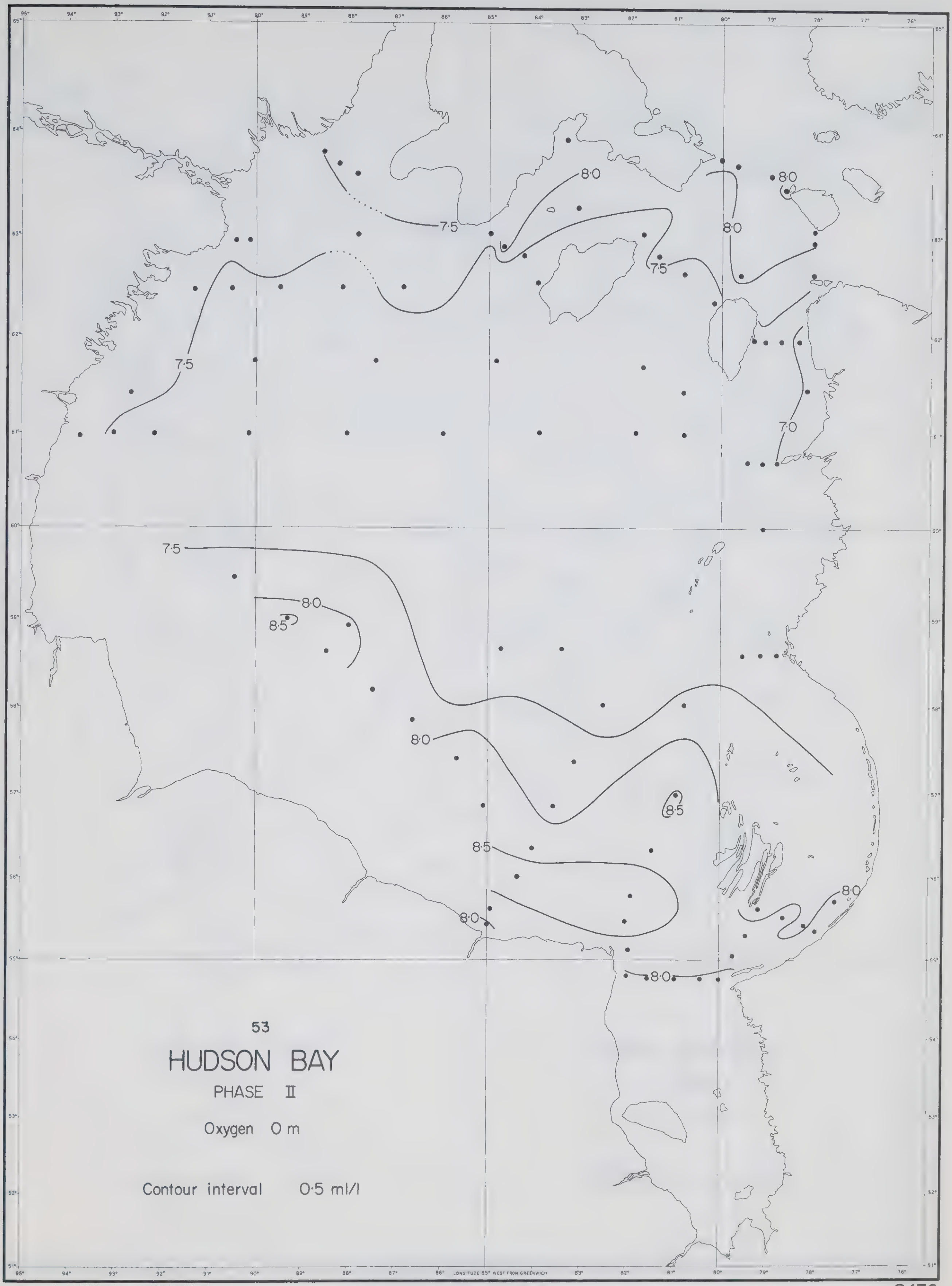
52

HUDSON BAY

PHASE II

Temperature deepest observation
Contour interval
< -1·0°C every 0·1 C°
> -1·0°C every 1·0 C°

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA



53

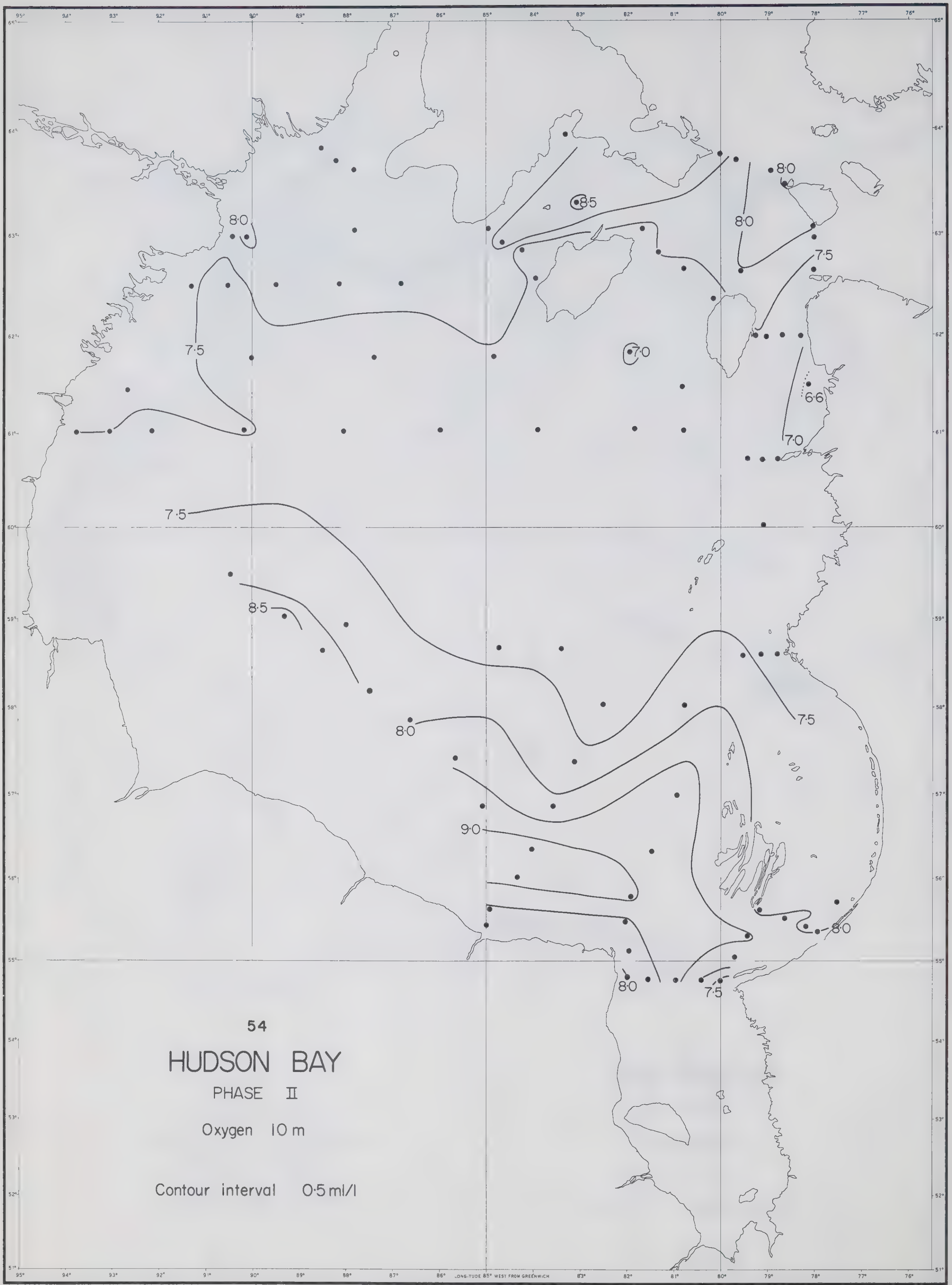
HUDSON BAY

PHASE II

Oxygen 0 m

Contour interval 0·5 ml/l

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA



54

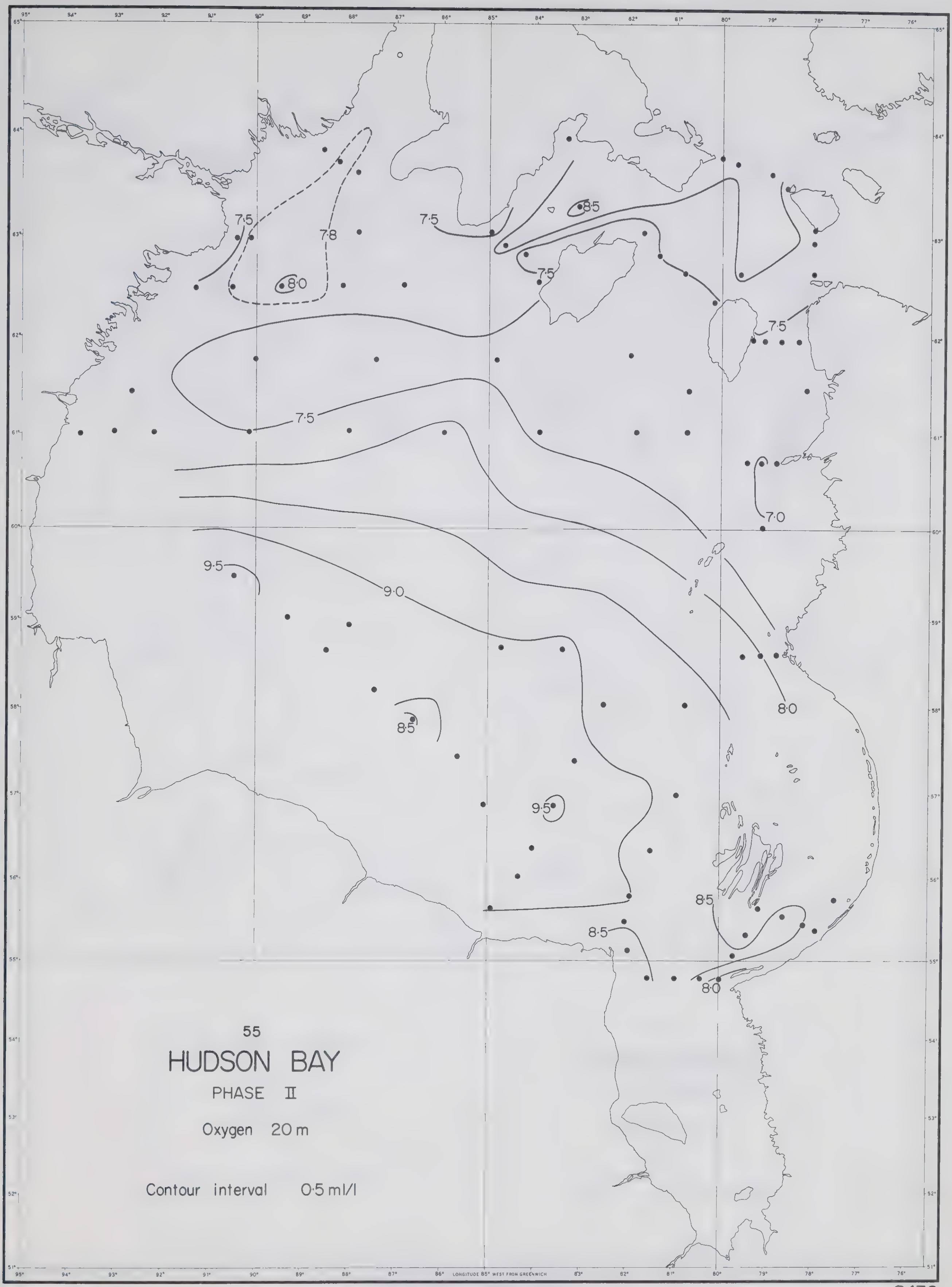
HUDSON BAY

PHASE II

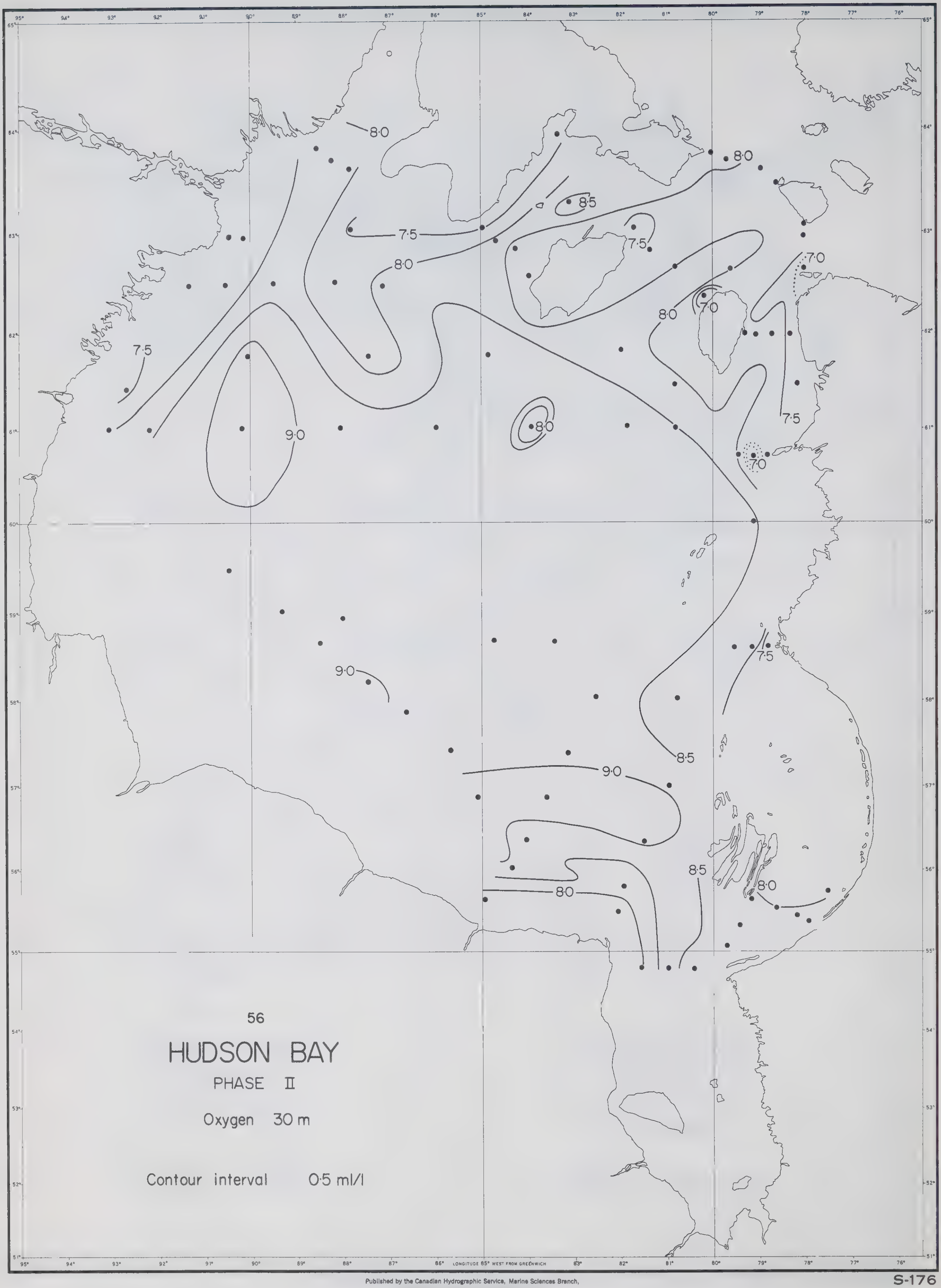
Oxygen 10 m

Contour interval 0·5 ml/l

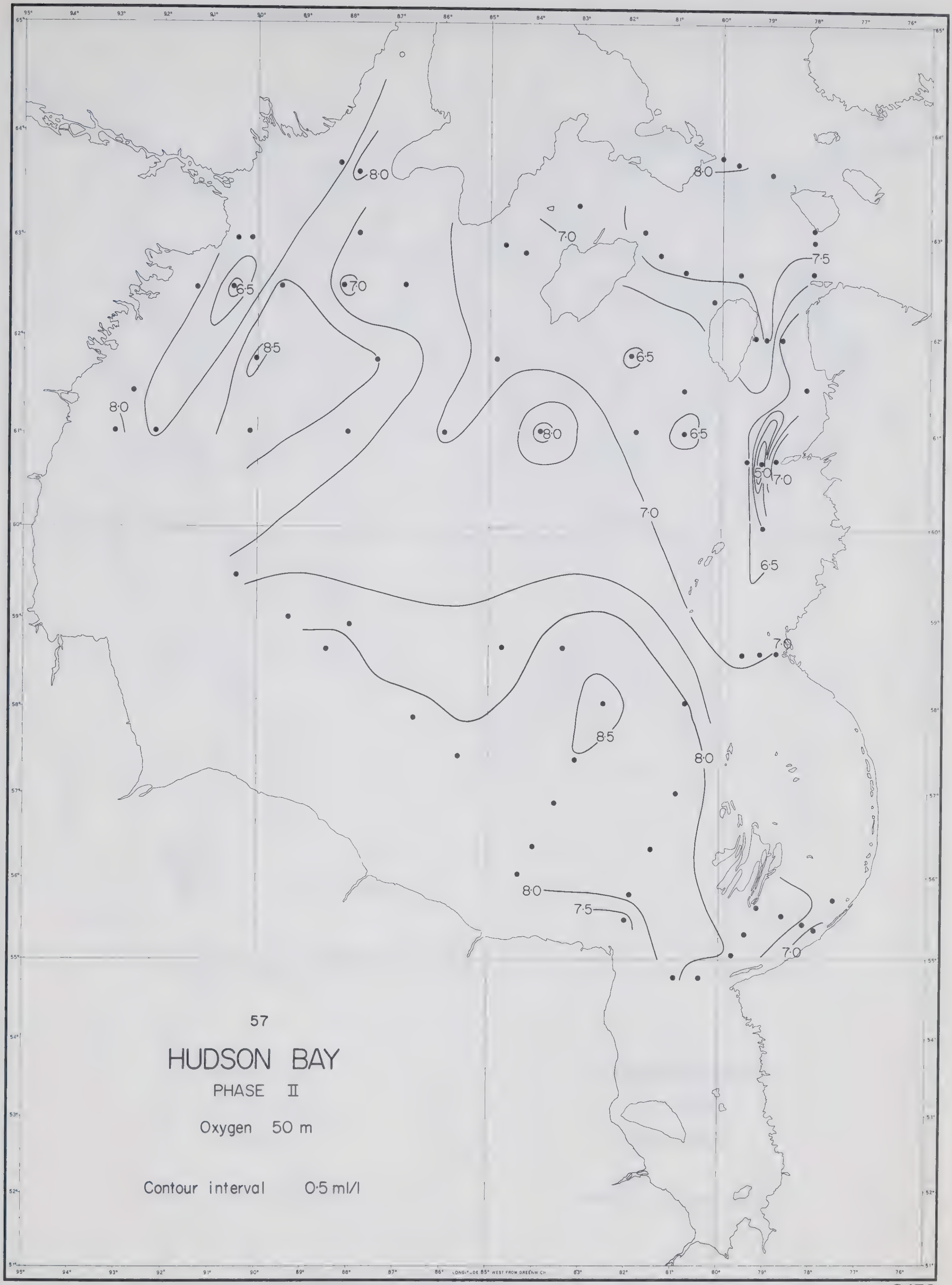
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA

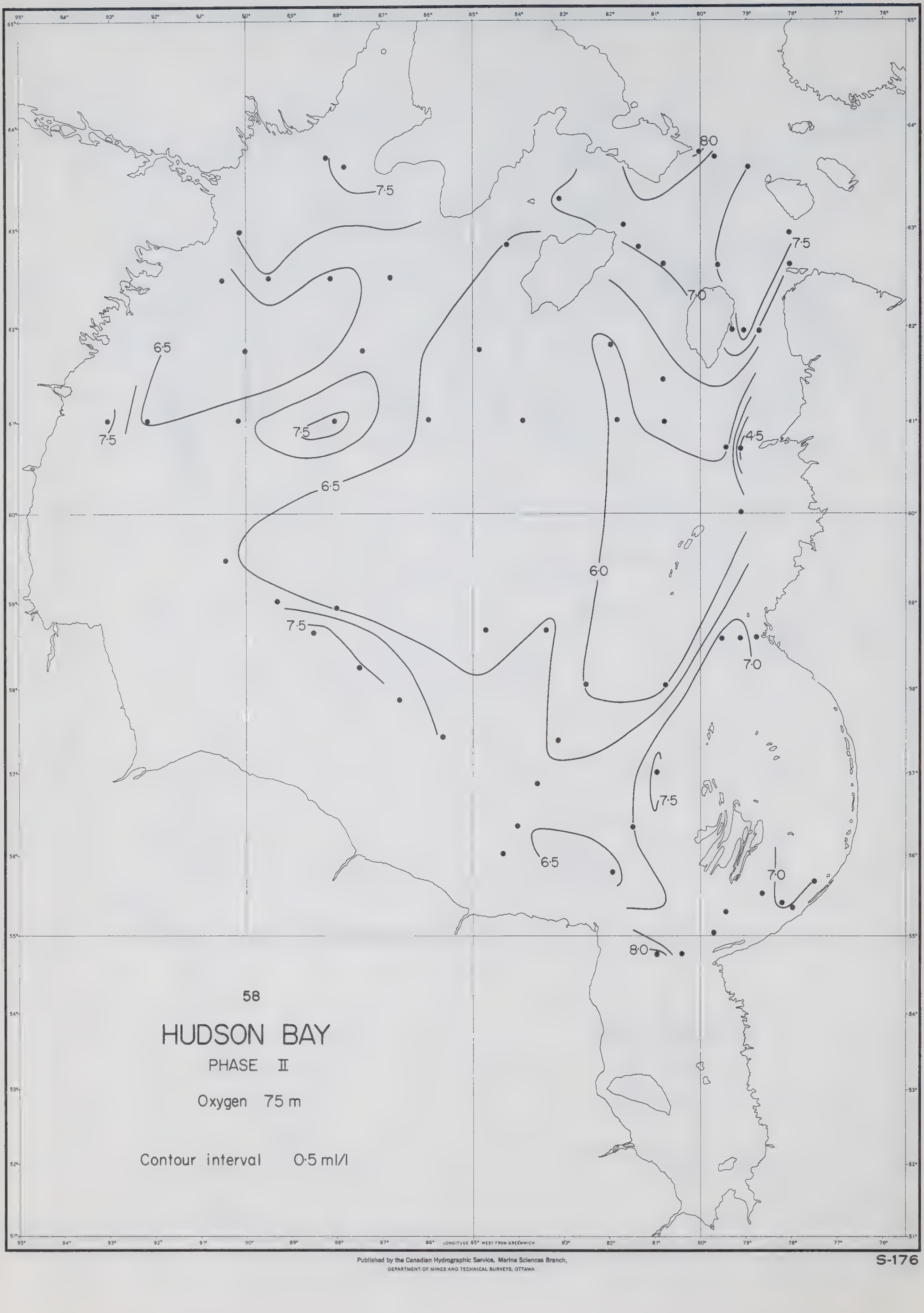


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



58

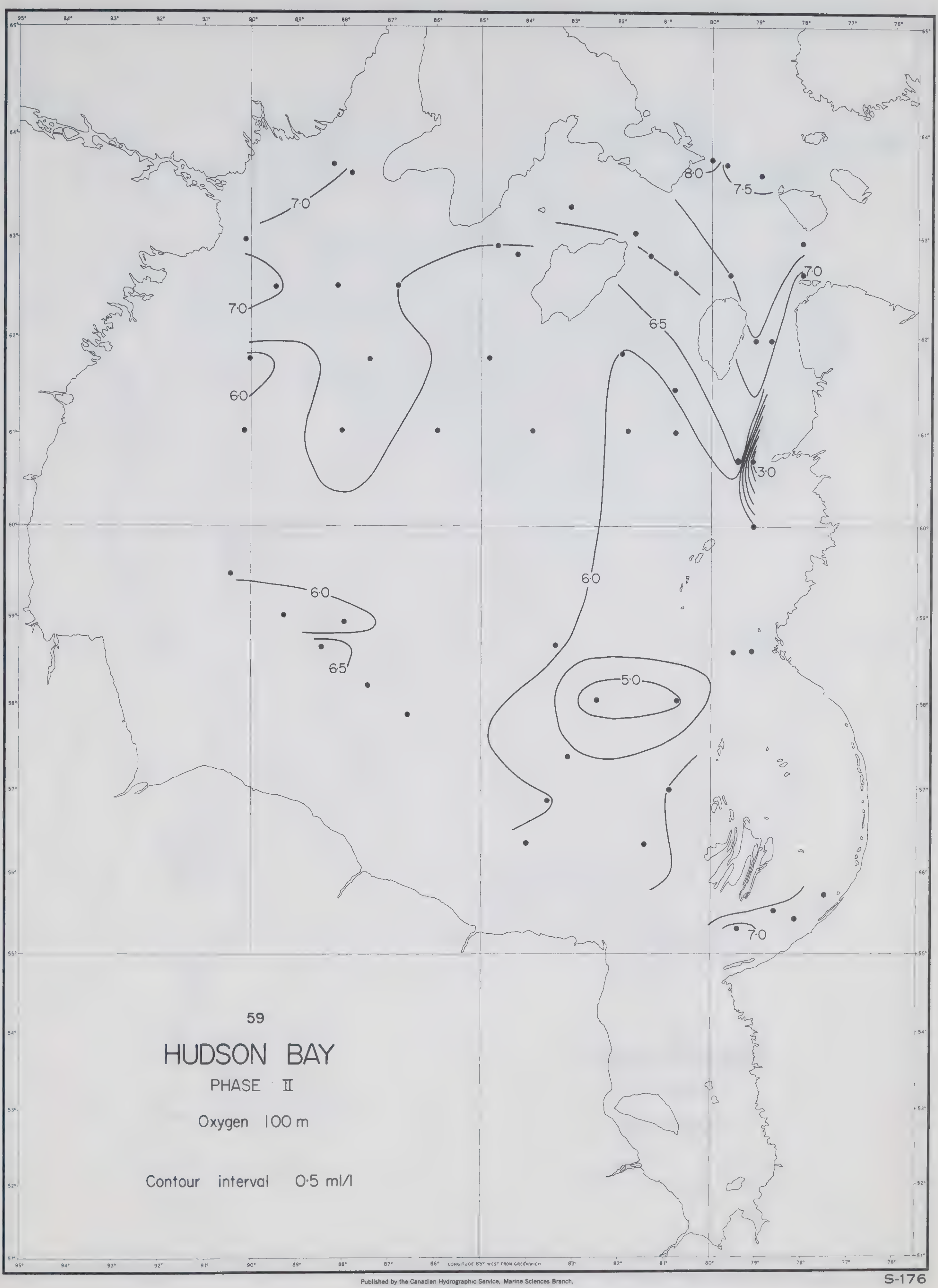
HUDSON BAY

PHASE II

Oxygen 75 m

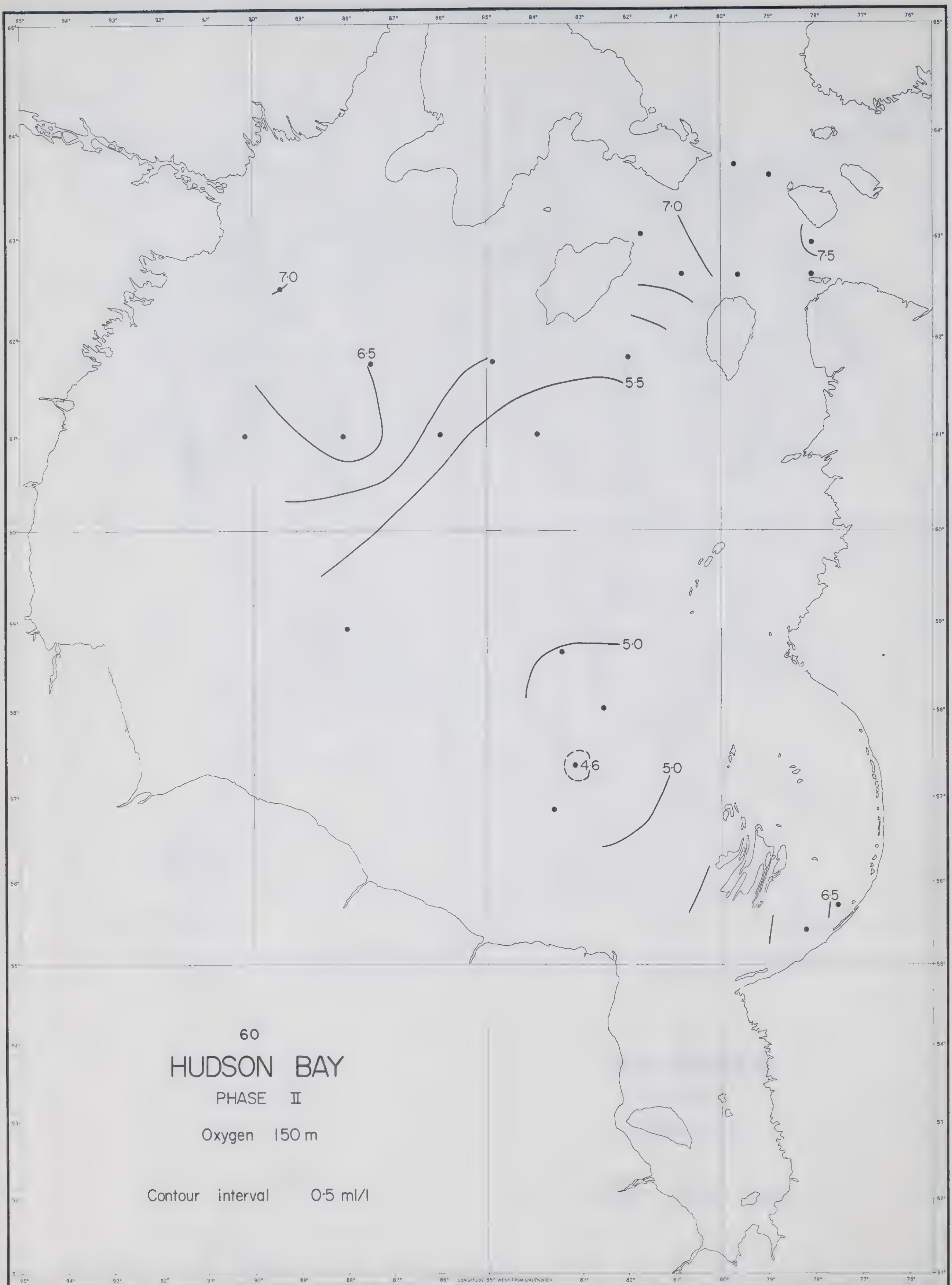
Contour interval 0·5 ml/l

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

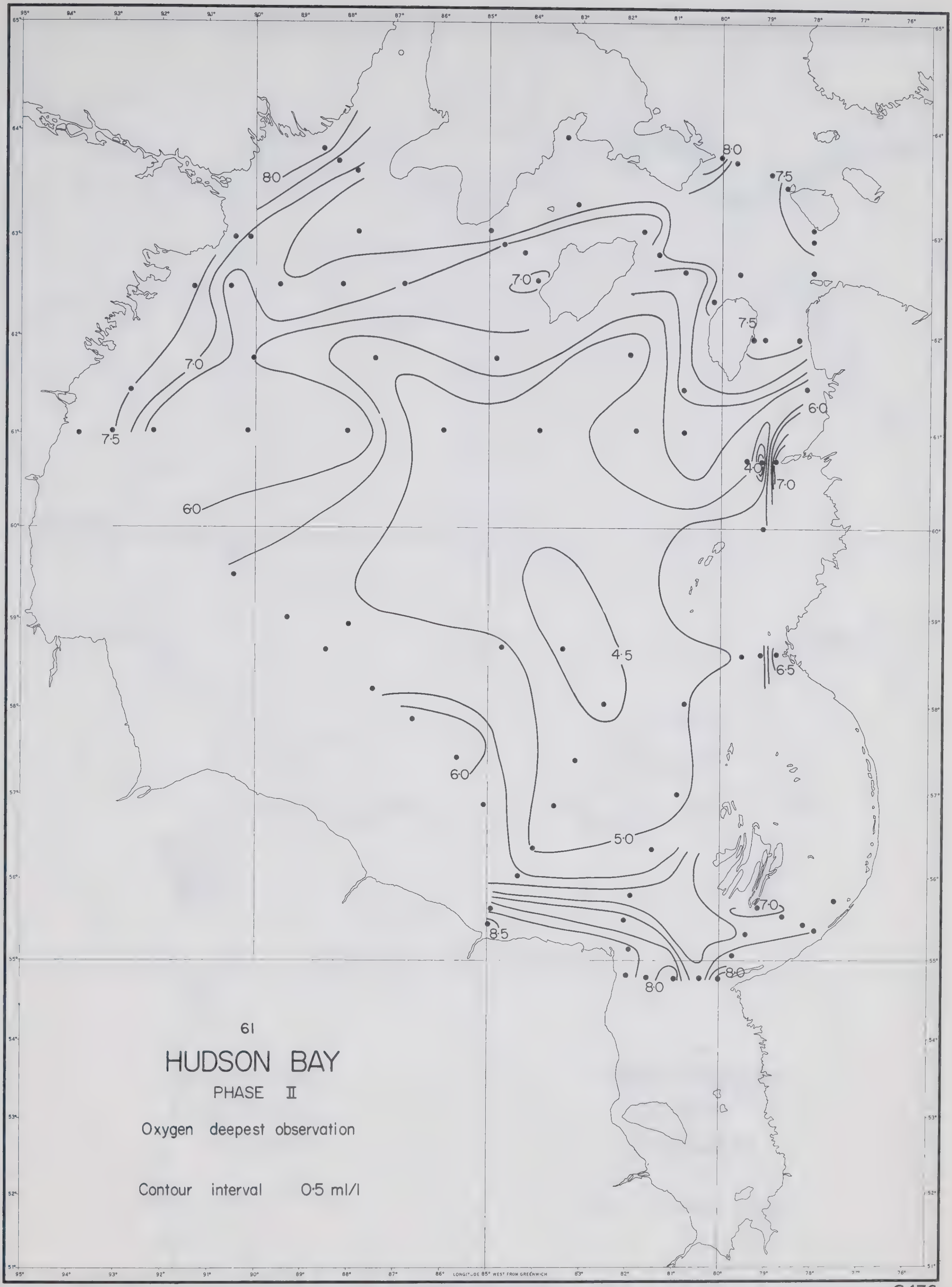


S-176

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA



61

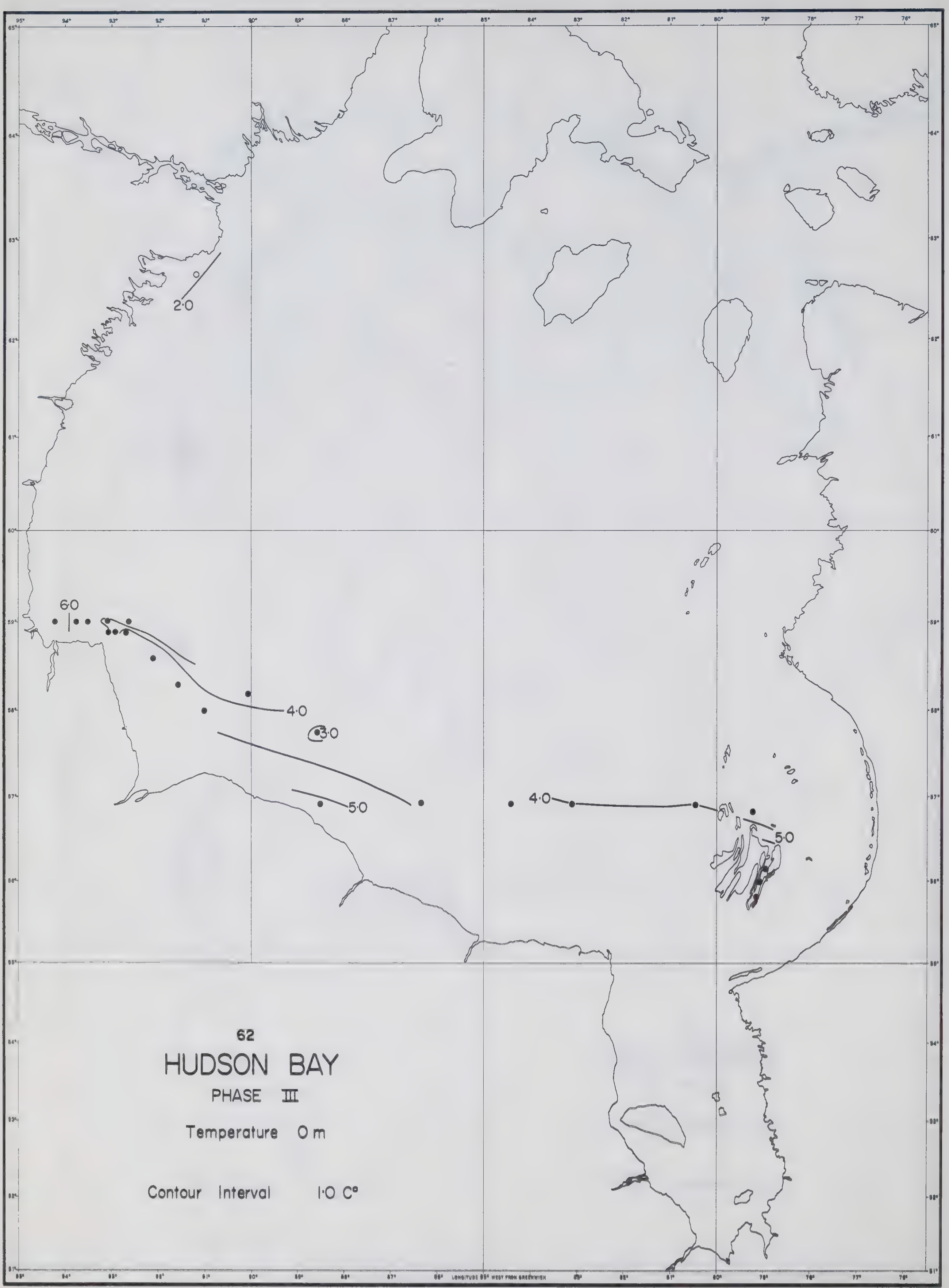
HUDSON BAY

PHASE II

Oxygen deepest observation

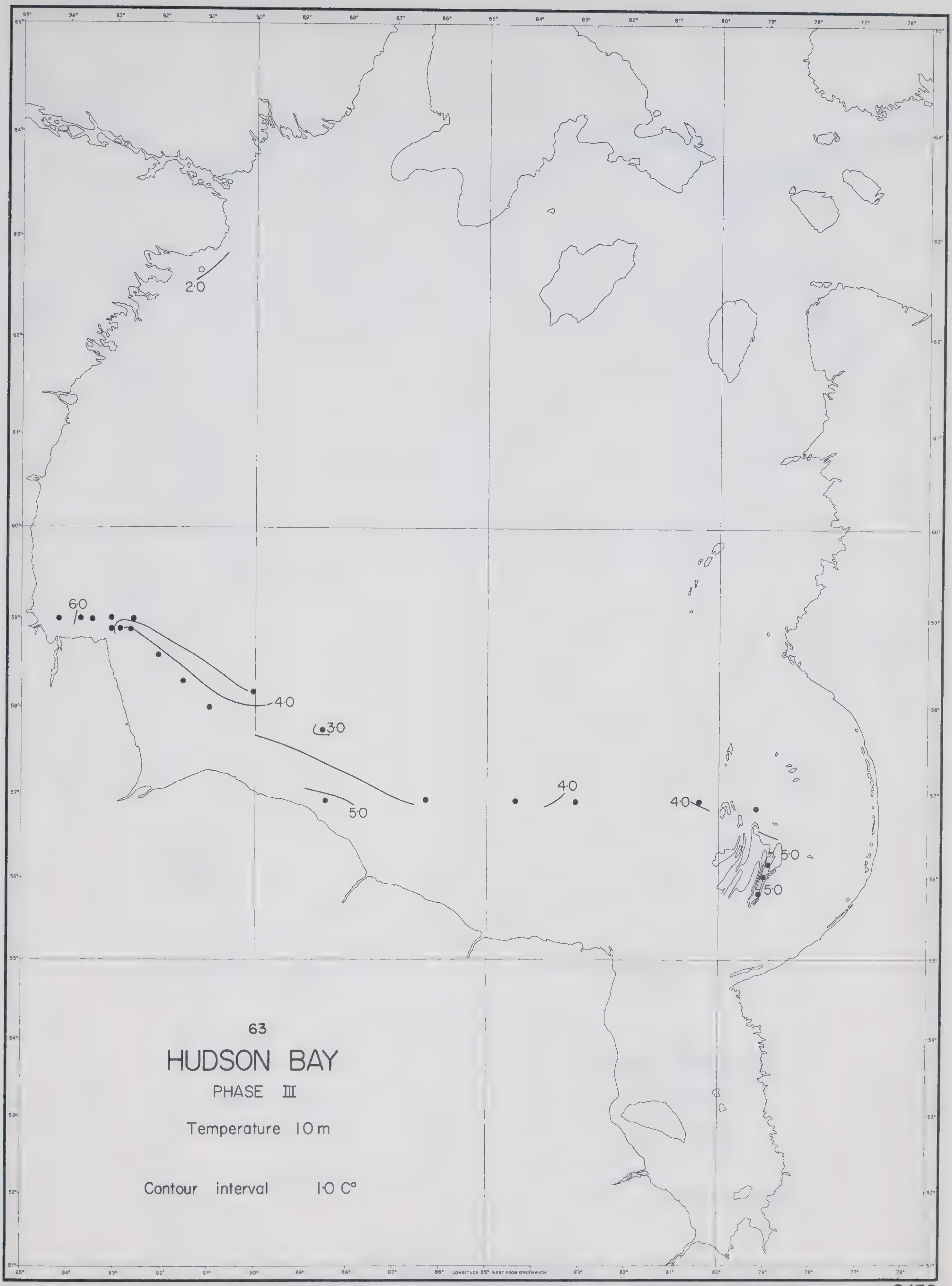
Contour interval 0·5 ml/l

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA

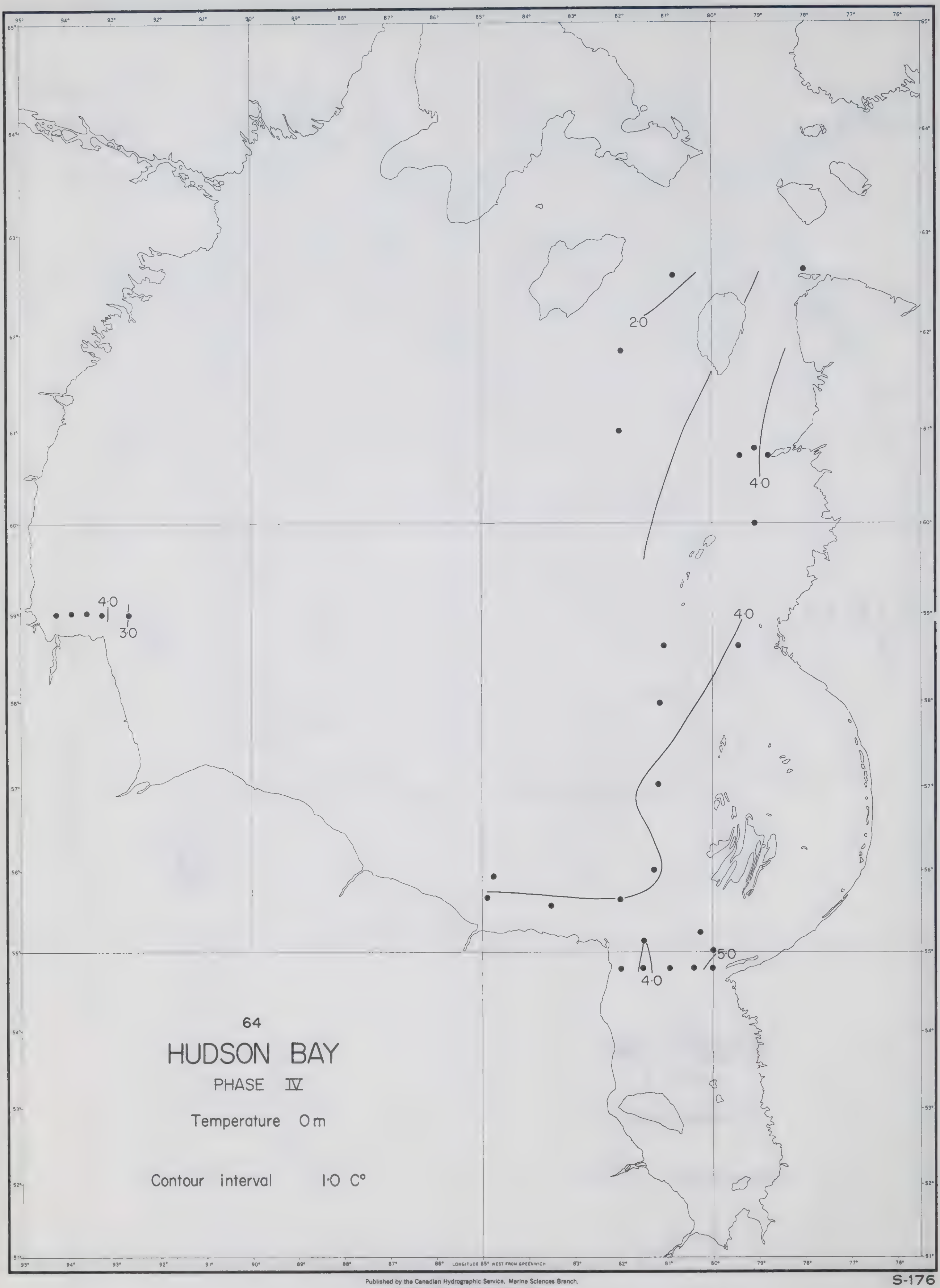


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176

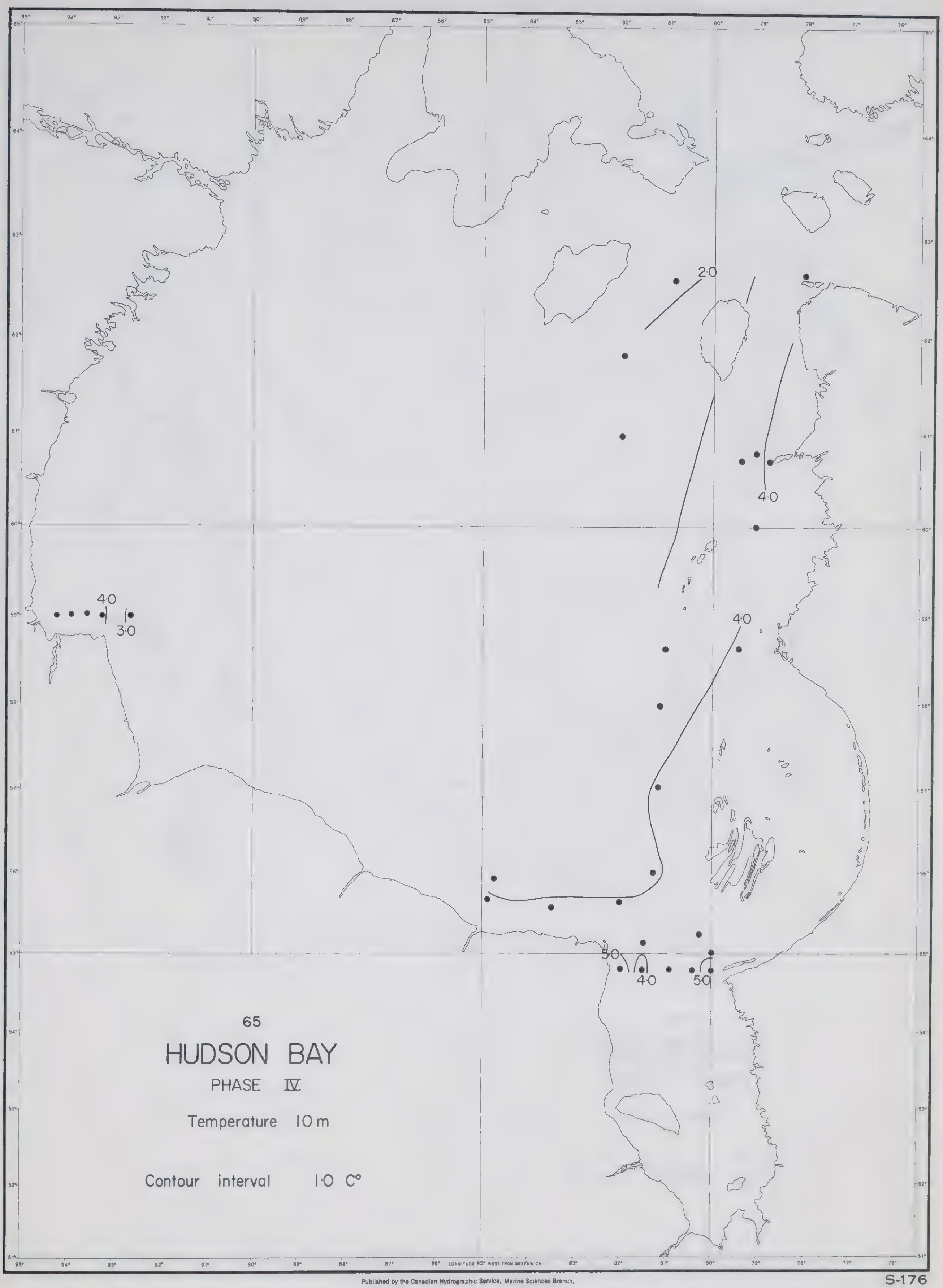


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA

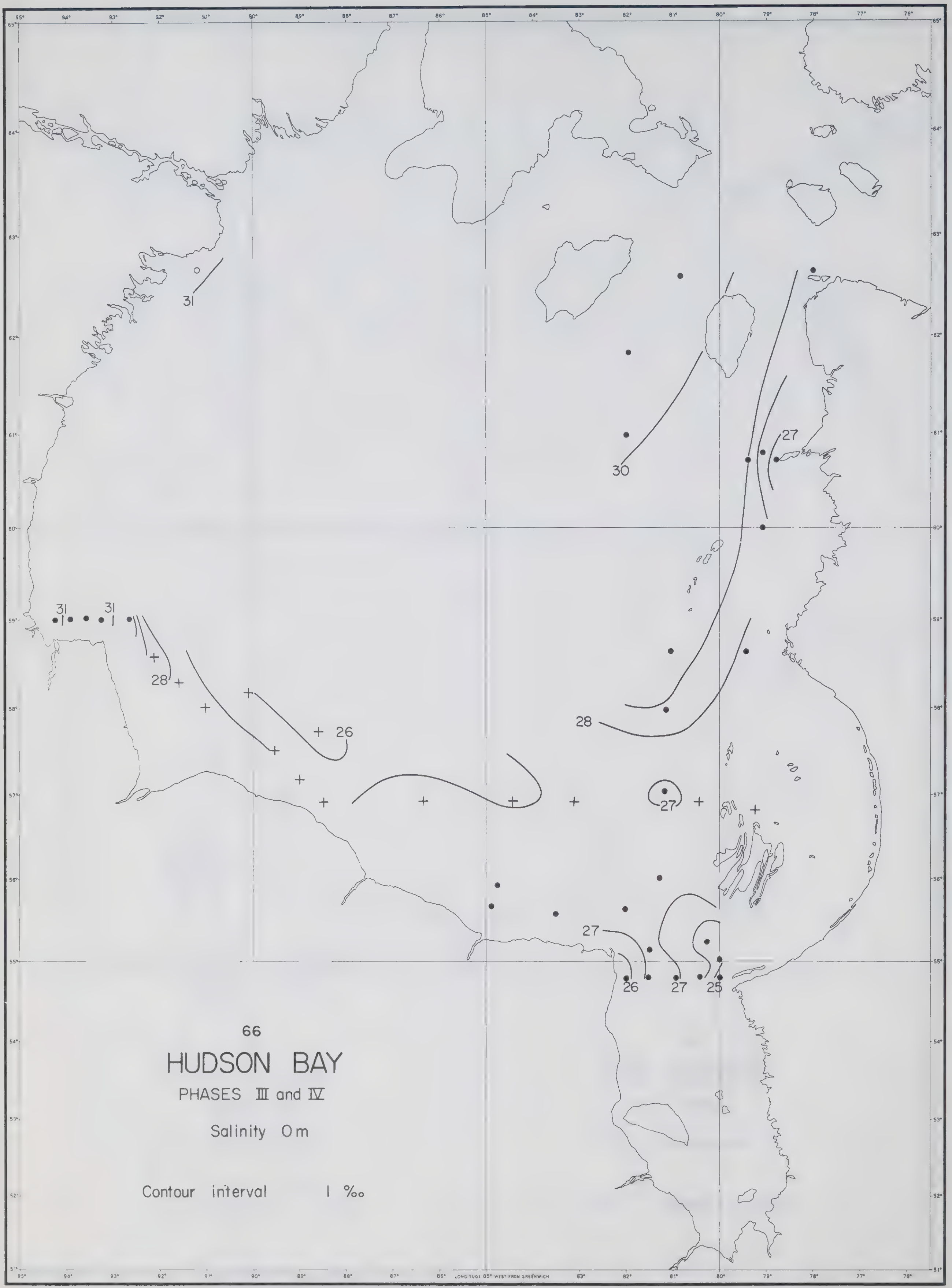


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176

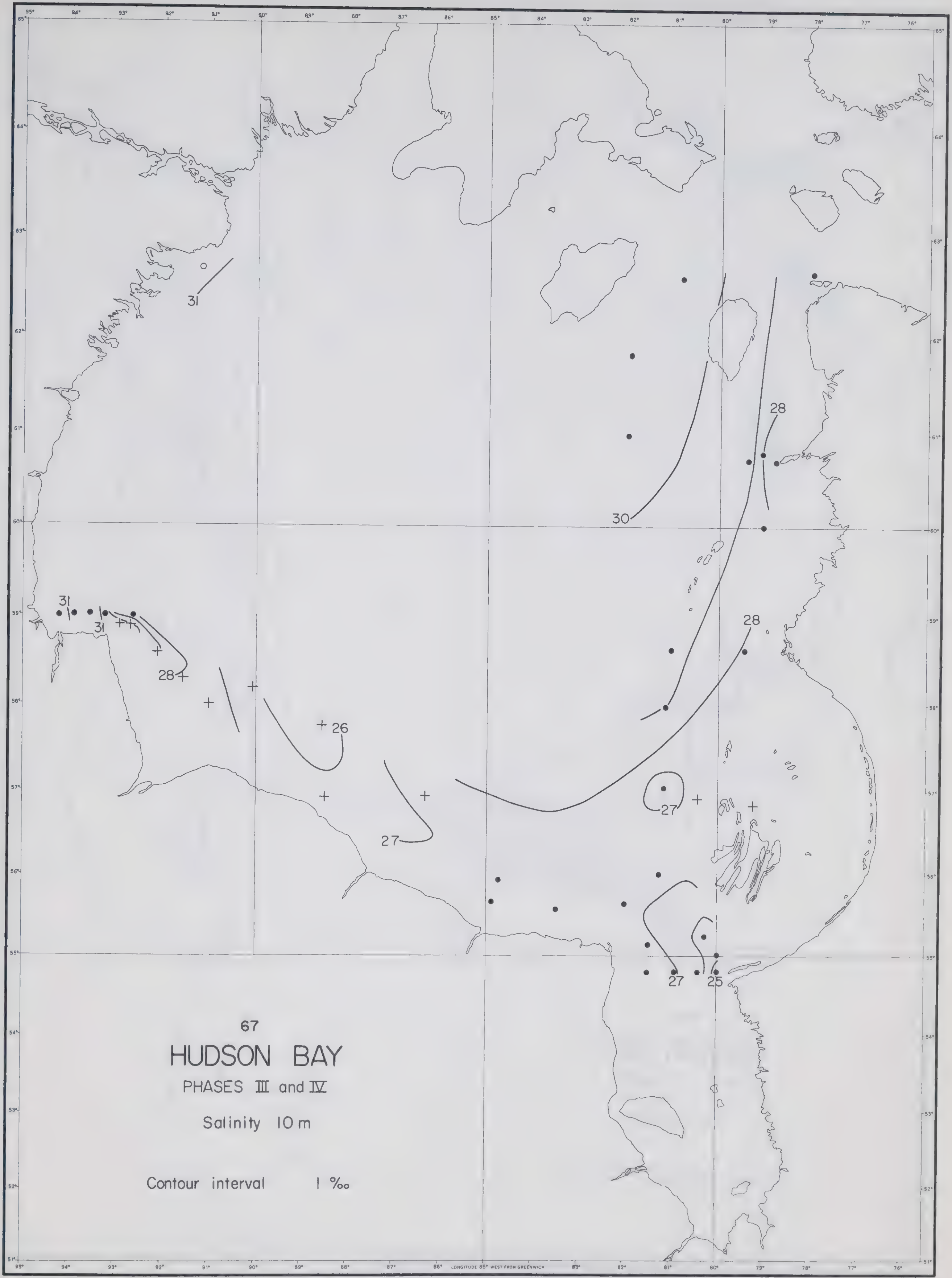


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



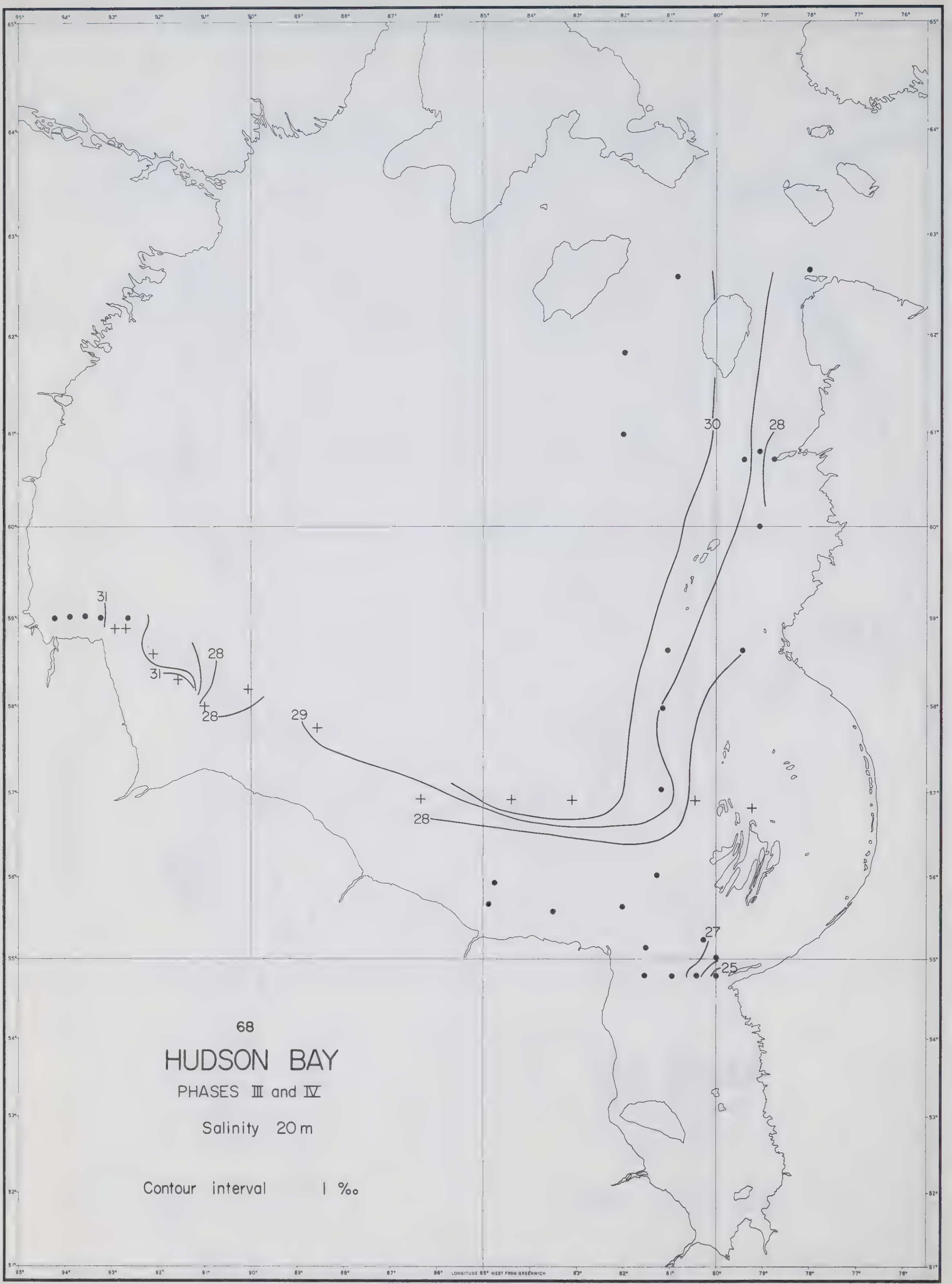
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



68

HUDSON BAY

PHASES III and IV

Salinity 20 m

Contour interval 1 ‰

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

69

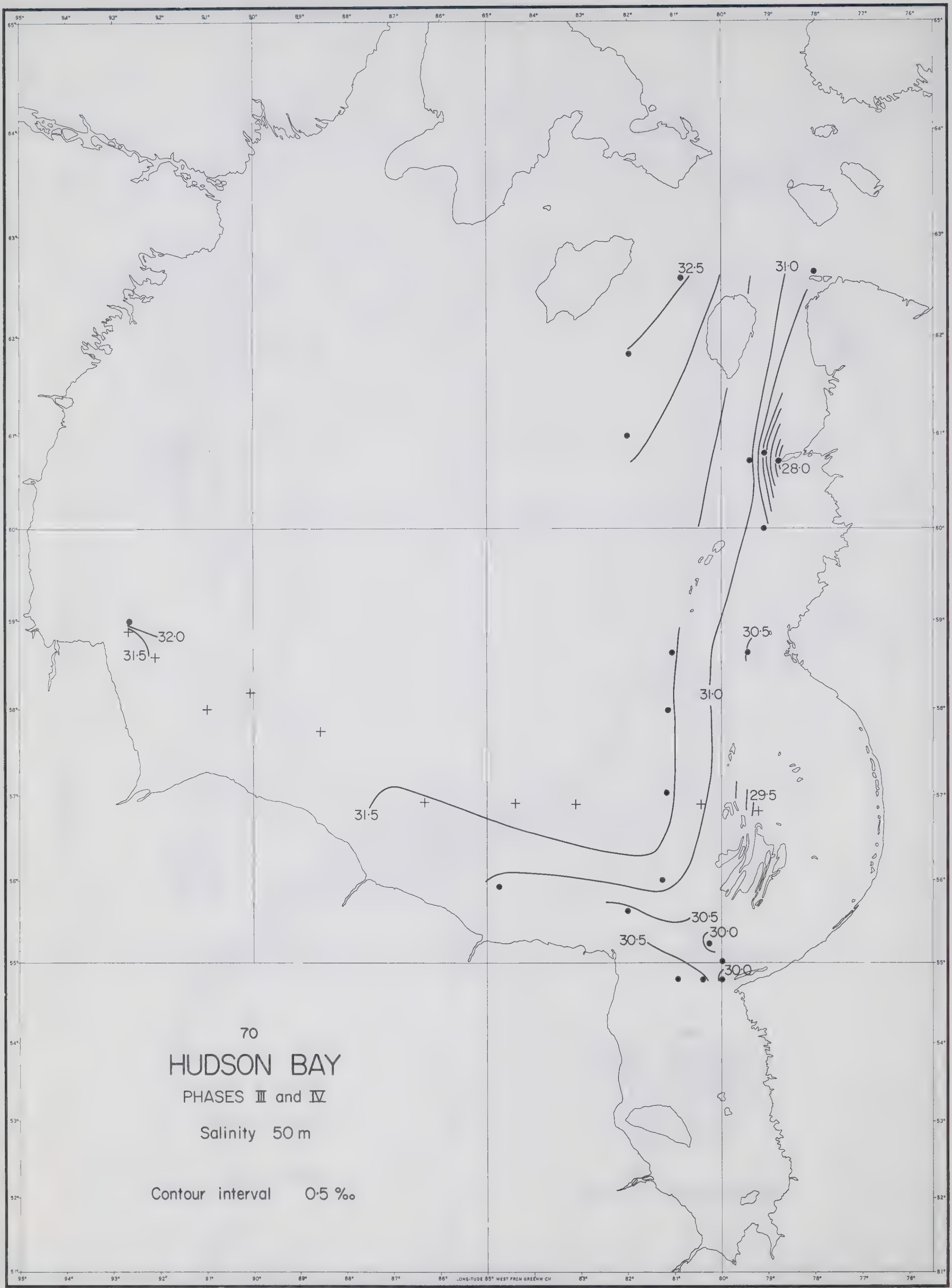
HUDSON BAY

PHASES III and IV

Salinity 30 m

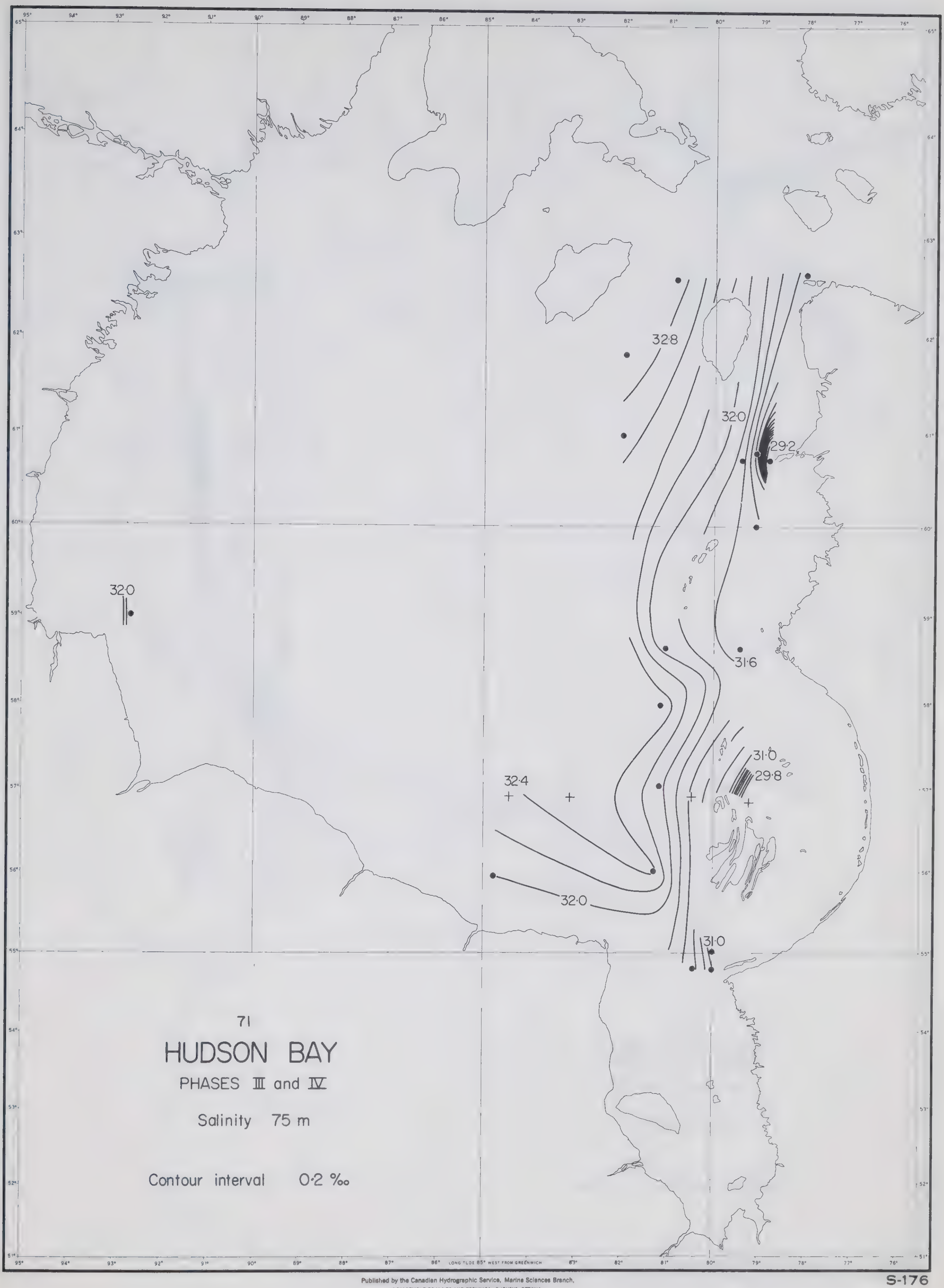
Contour interval 0·5 ‰

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

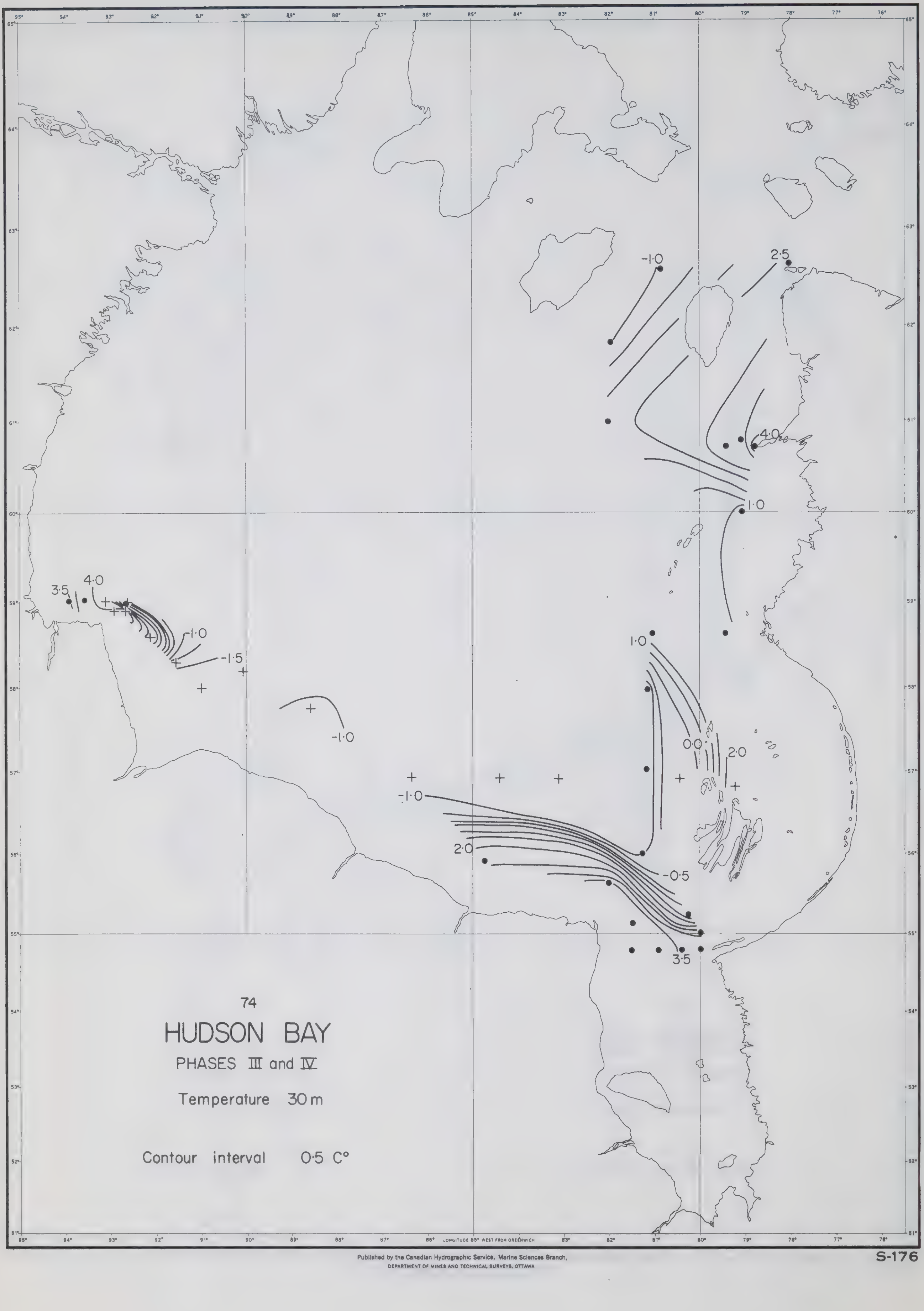


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176

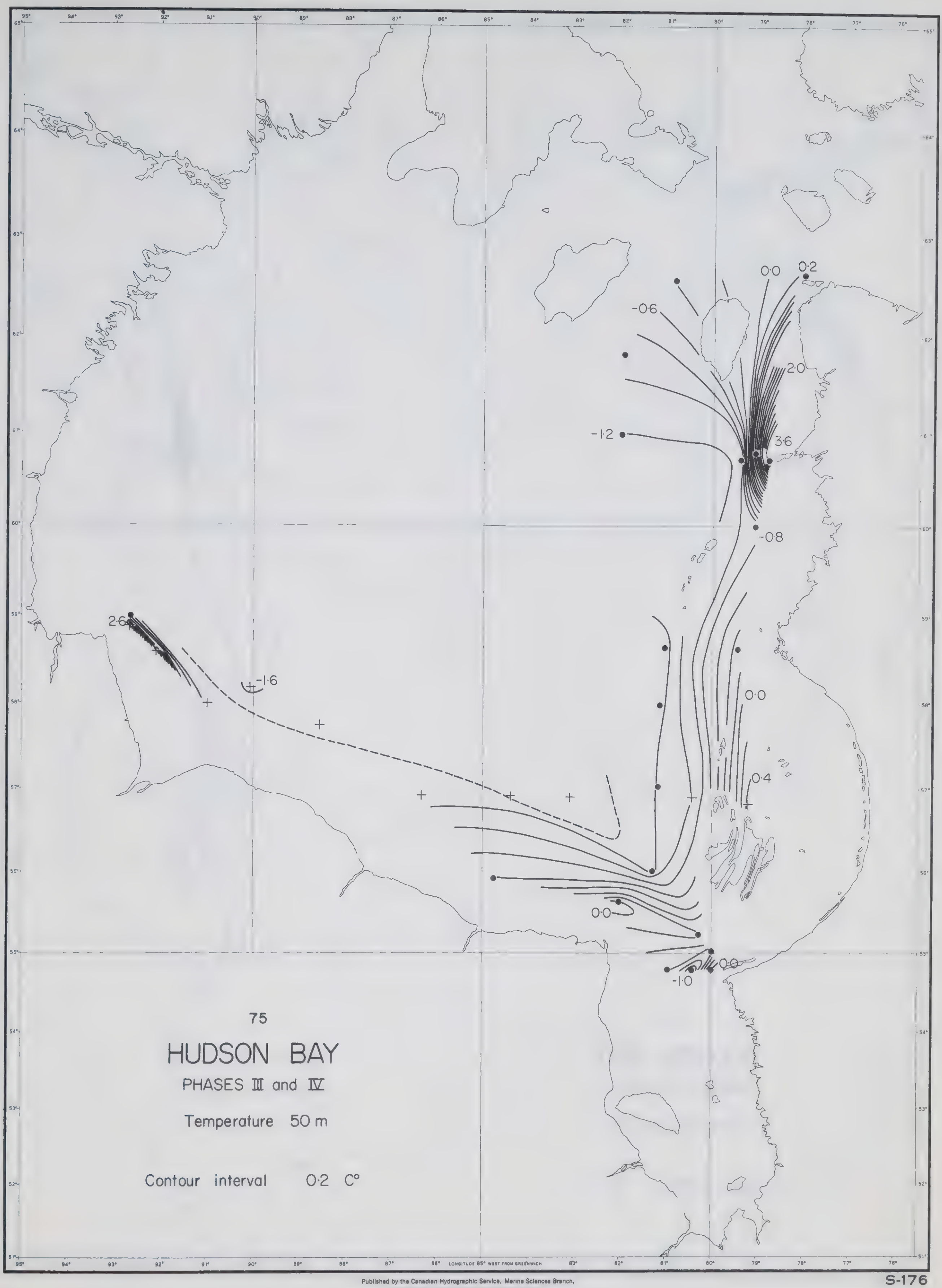


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



75

HUDSON BAY

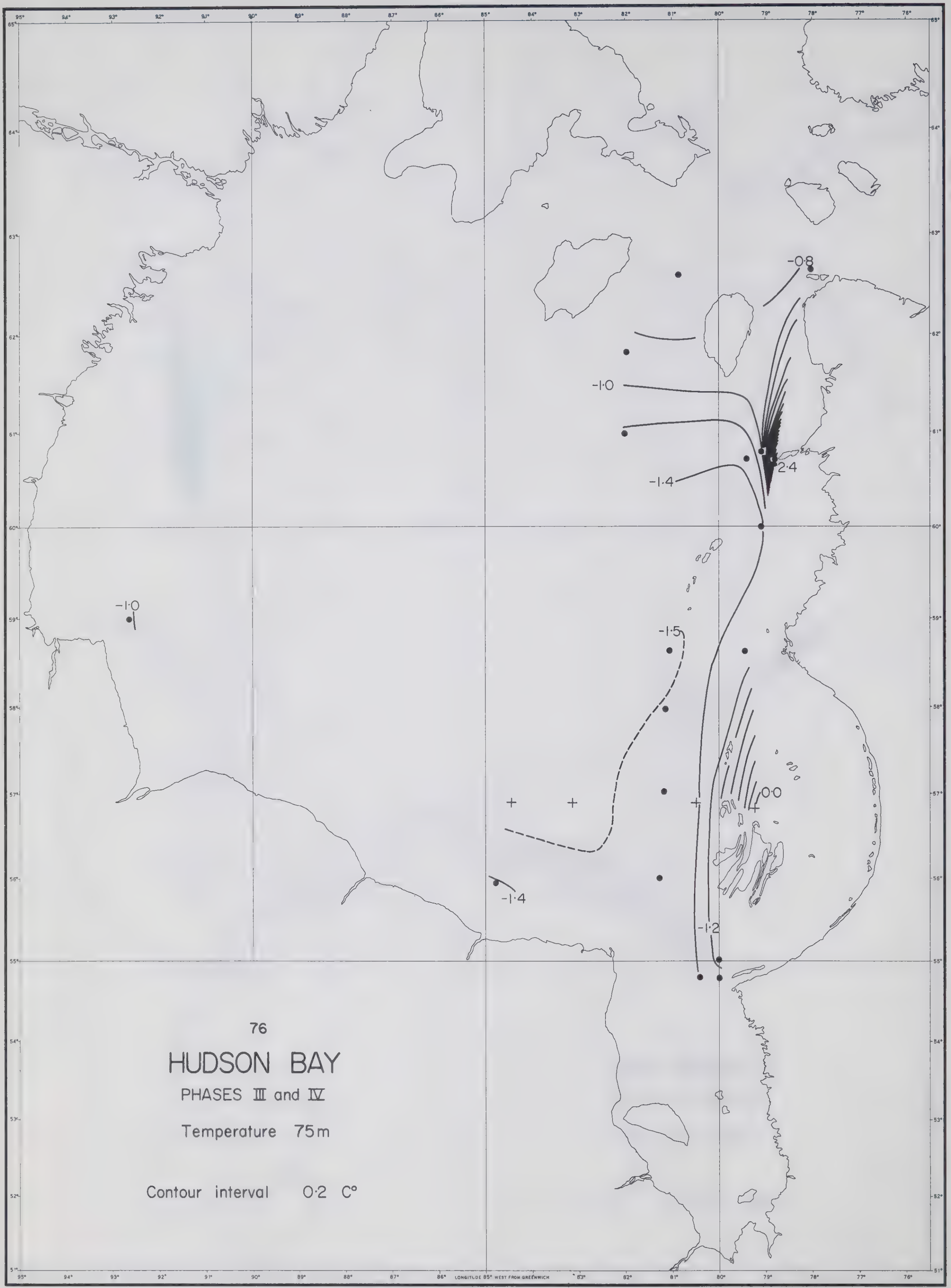
PHASES III and IV

Temperature 50 m

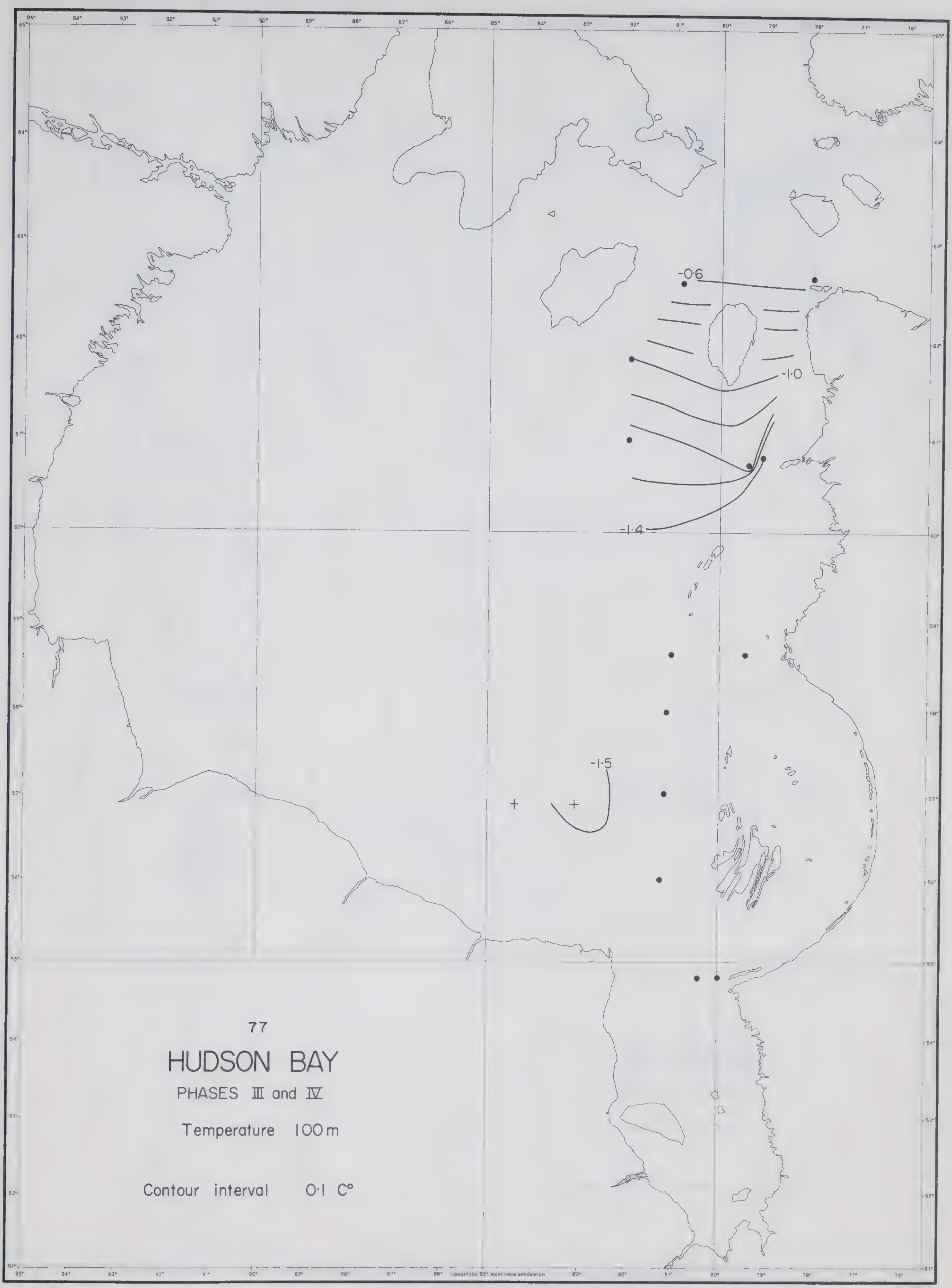
Contour interval 0·2 C°

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176

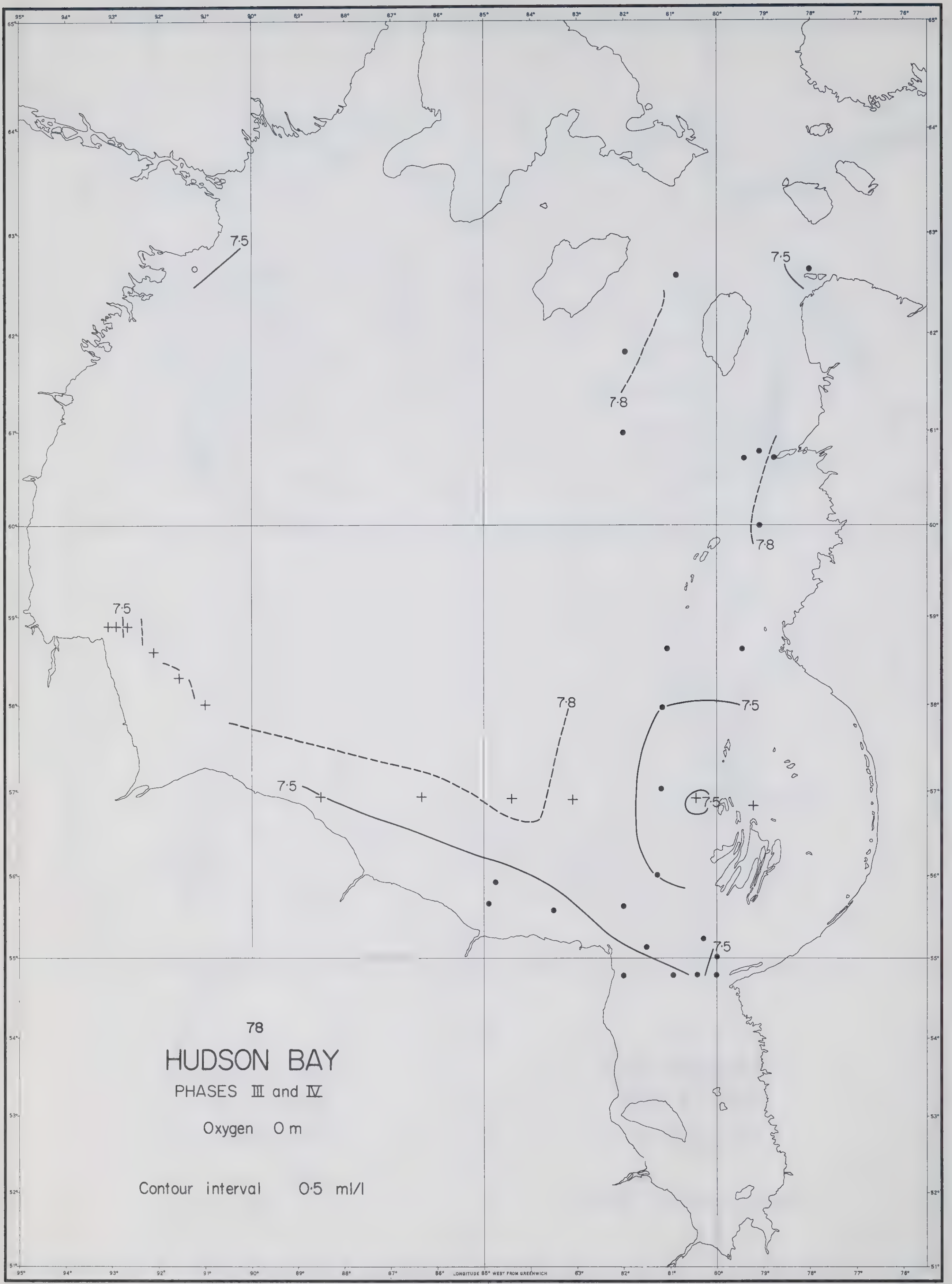


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



78

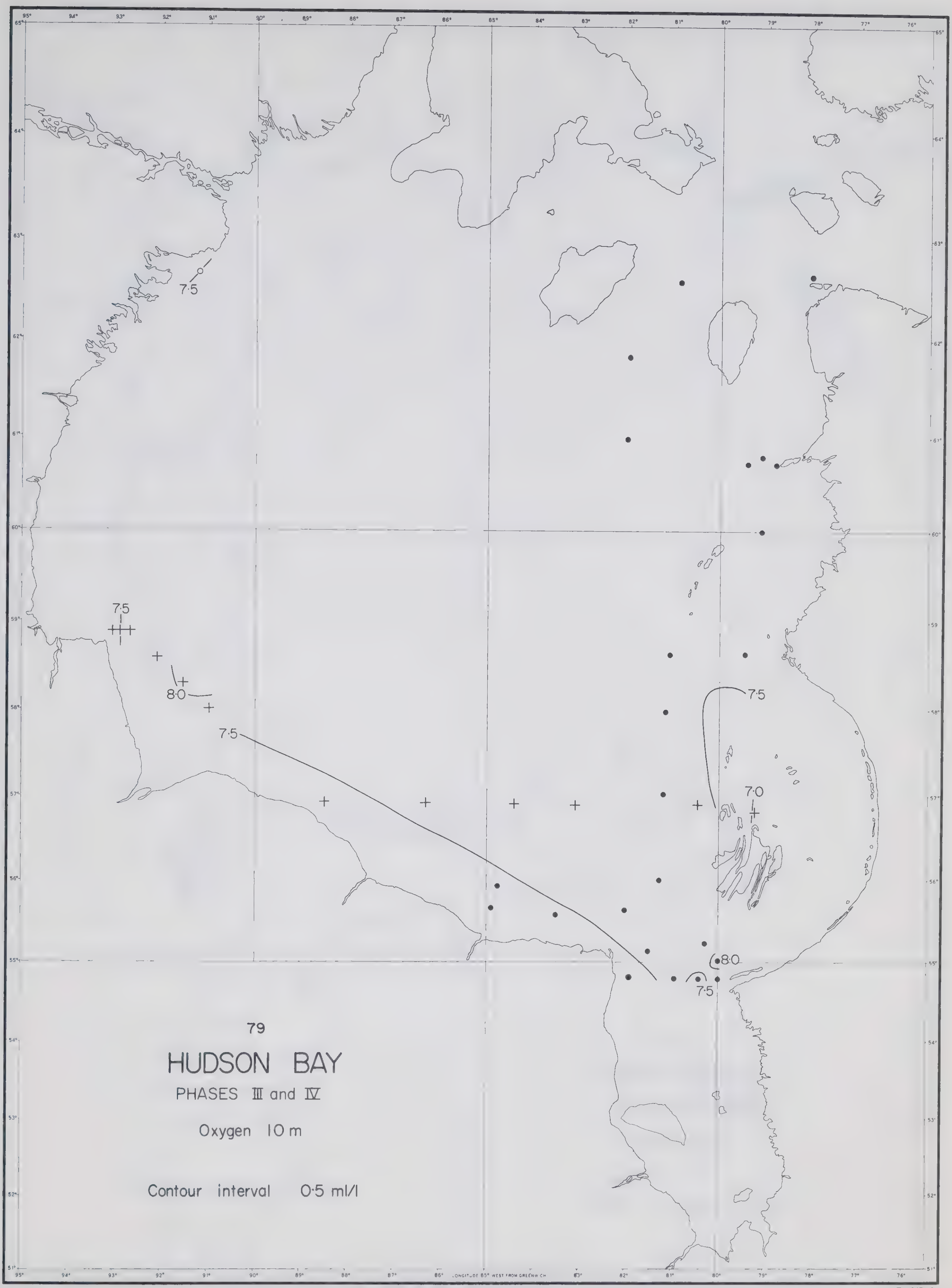
HUDSON BAY

PHASES III and IV

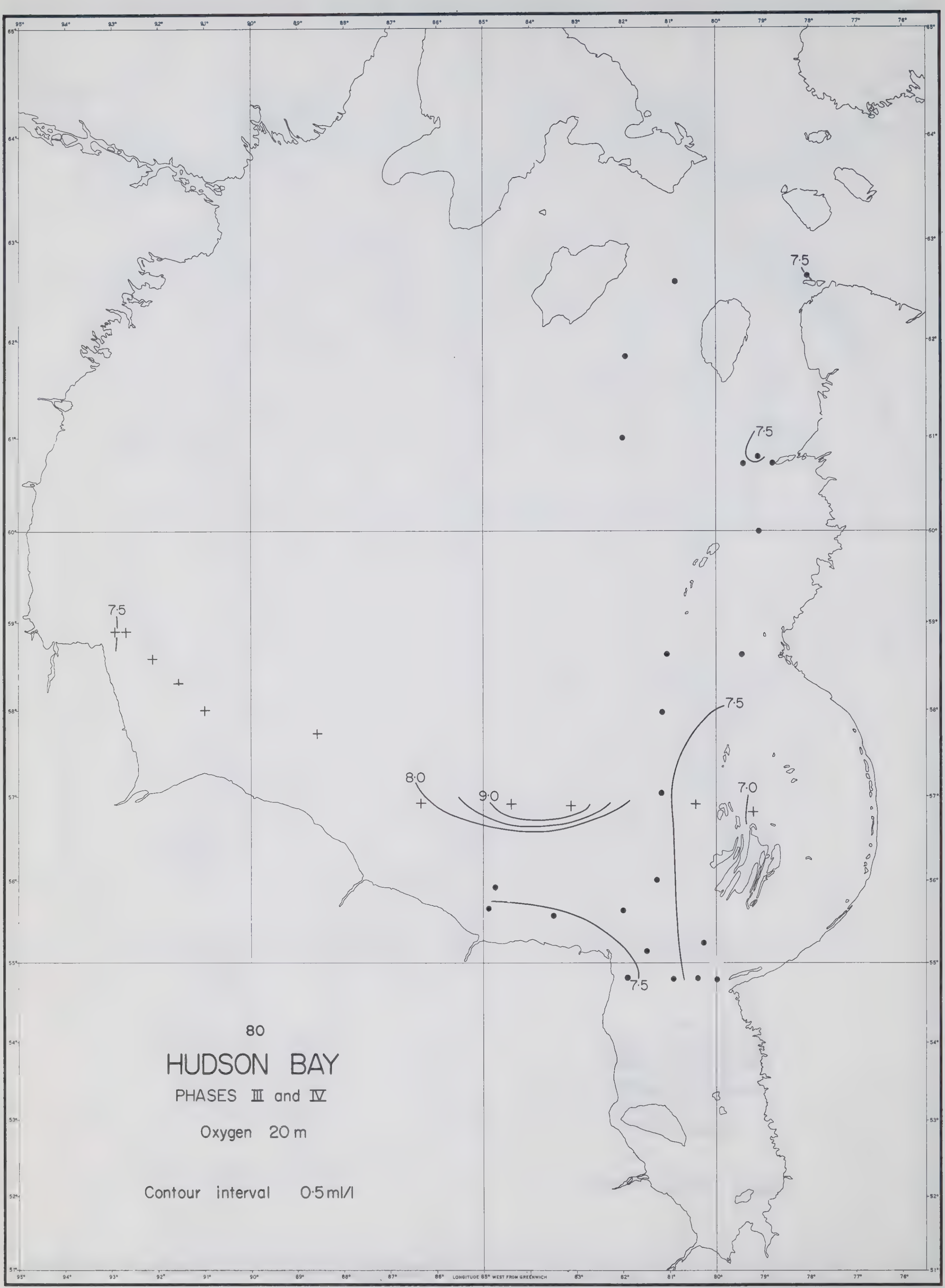
Oxygen 0 m

Contour interval 0·5 ml/l

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

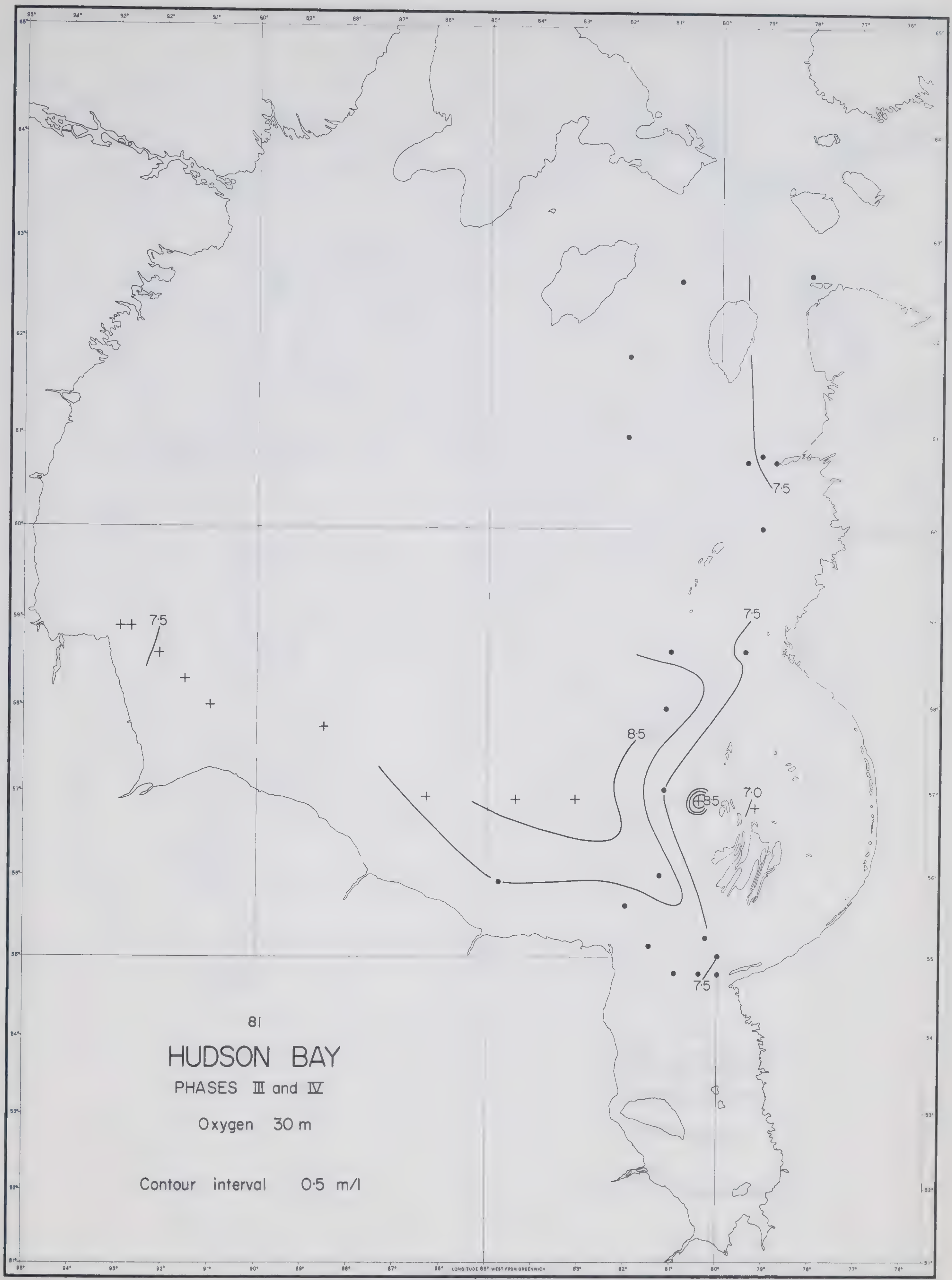


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA



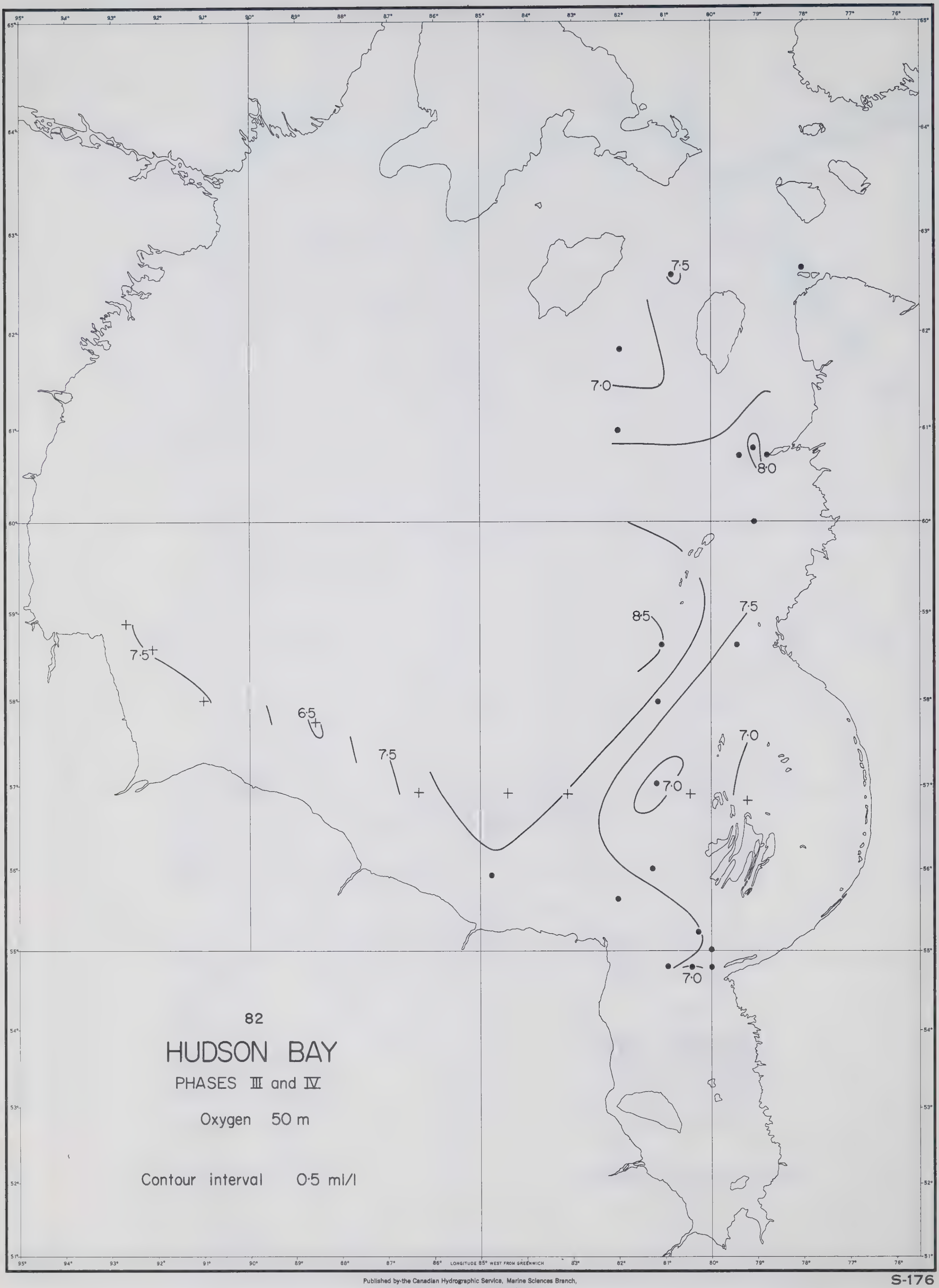
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176

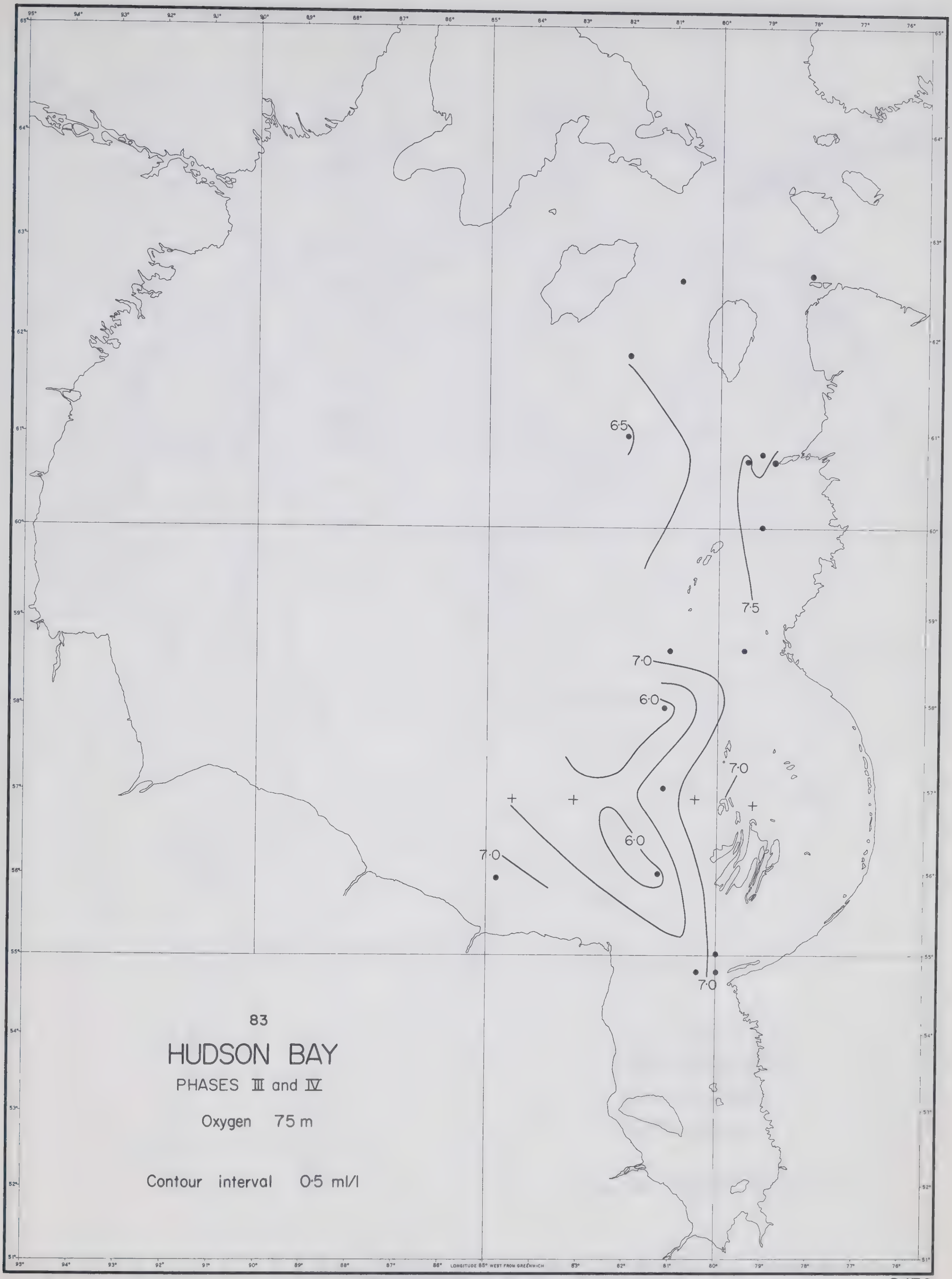


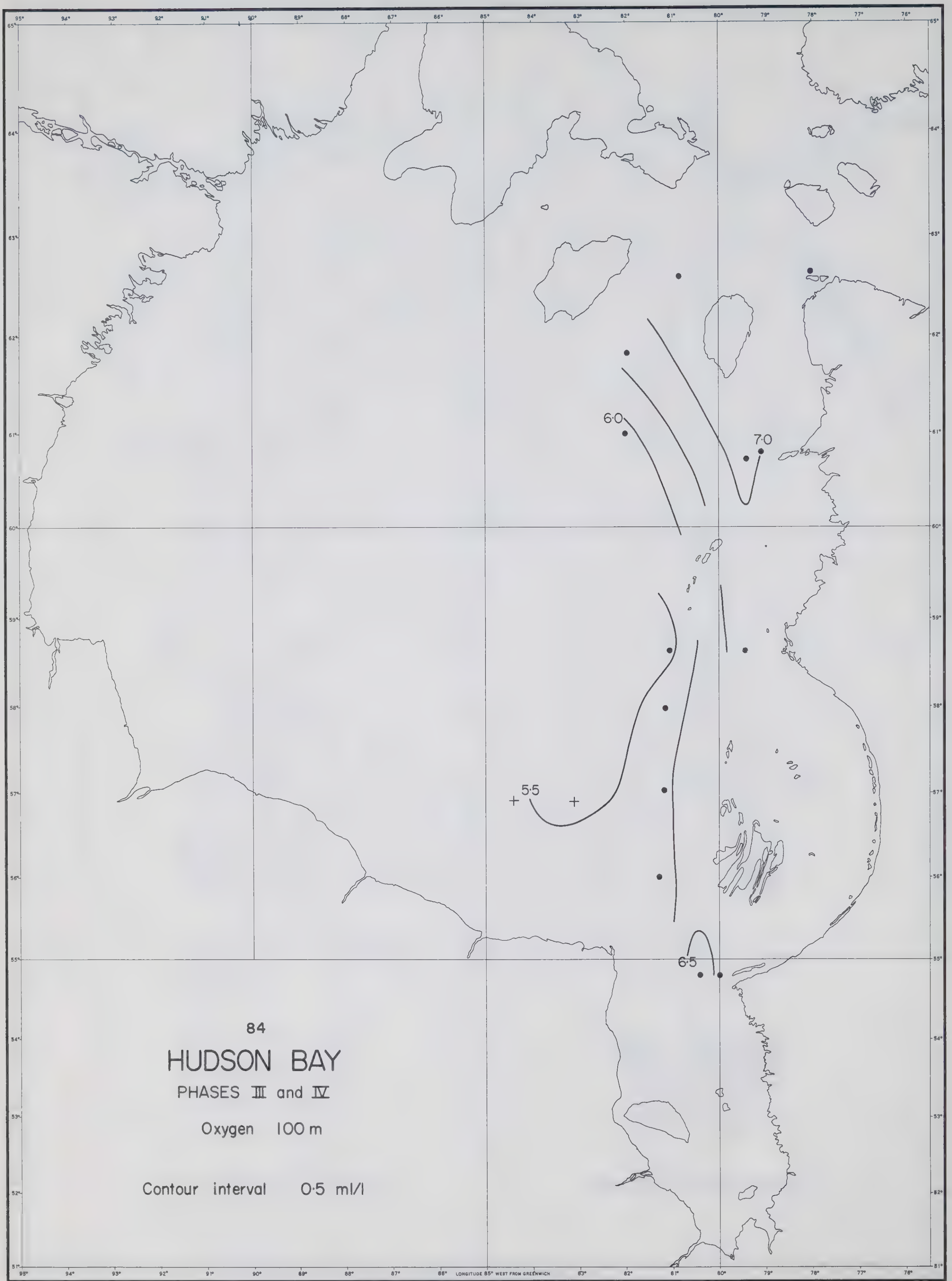
Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA


83
HUDSON BAY
PHASES III and IV
Oxygen 75 m
Contour interval 0·5 ml/l
6·5
7·5
7·0
6·0
7·0
6·0
7·0
7·0
LONGITUDE 85° WEST FROM GREENWICH



84

HUDSON BAY

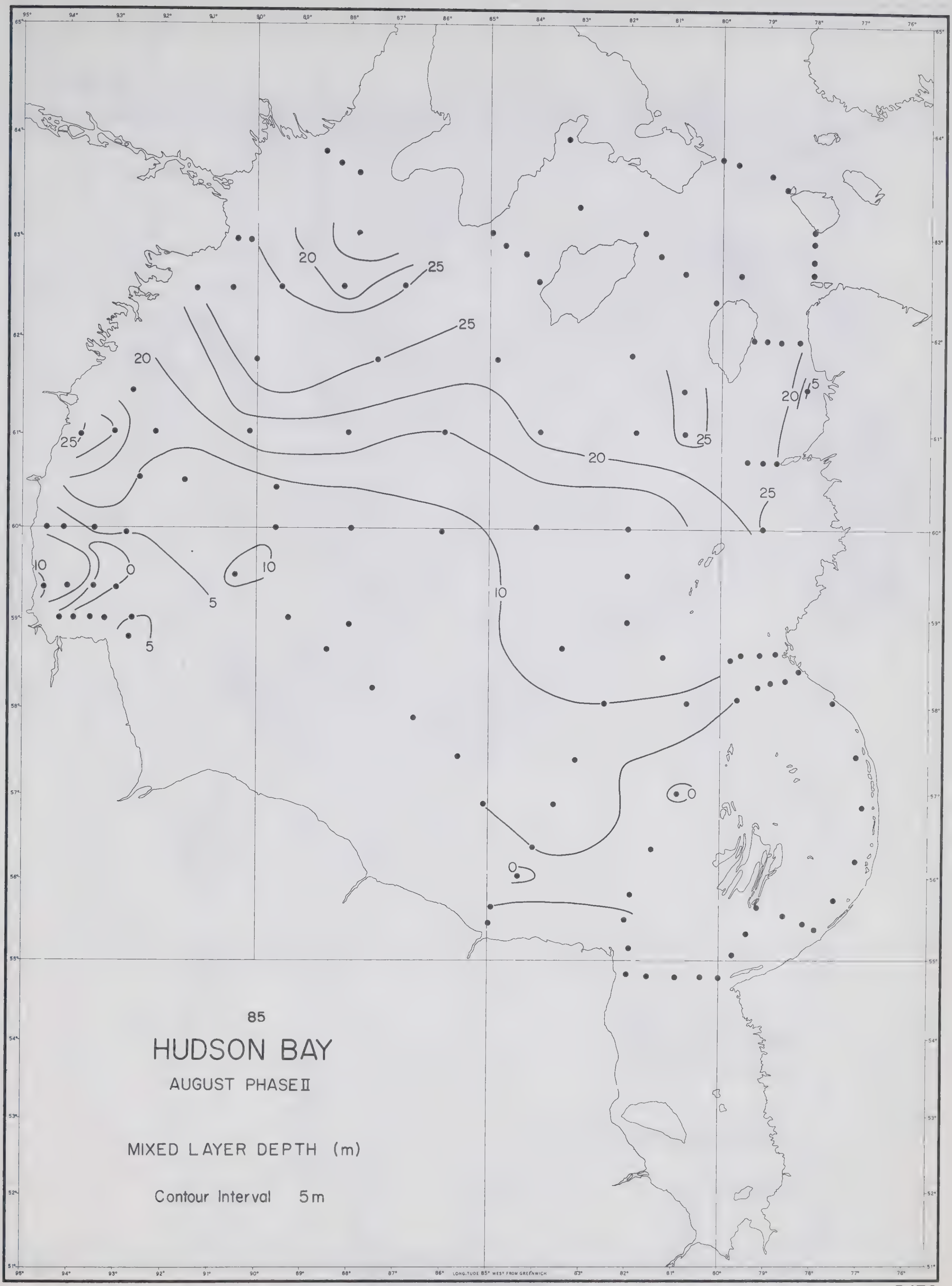
PHASES III and IV

Oxygen 100 m

Contour interval 0·5 ml/l

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



85

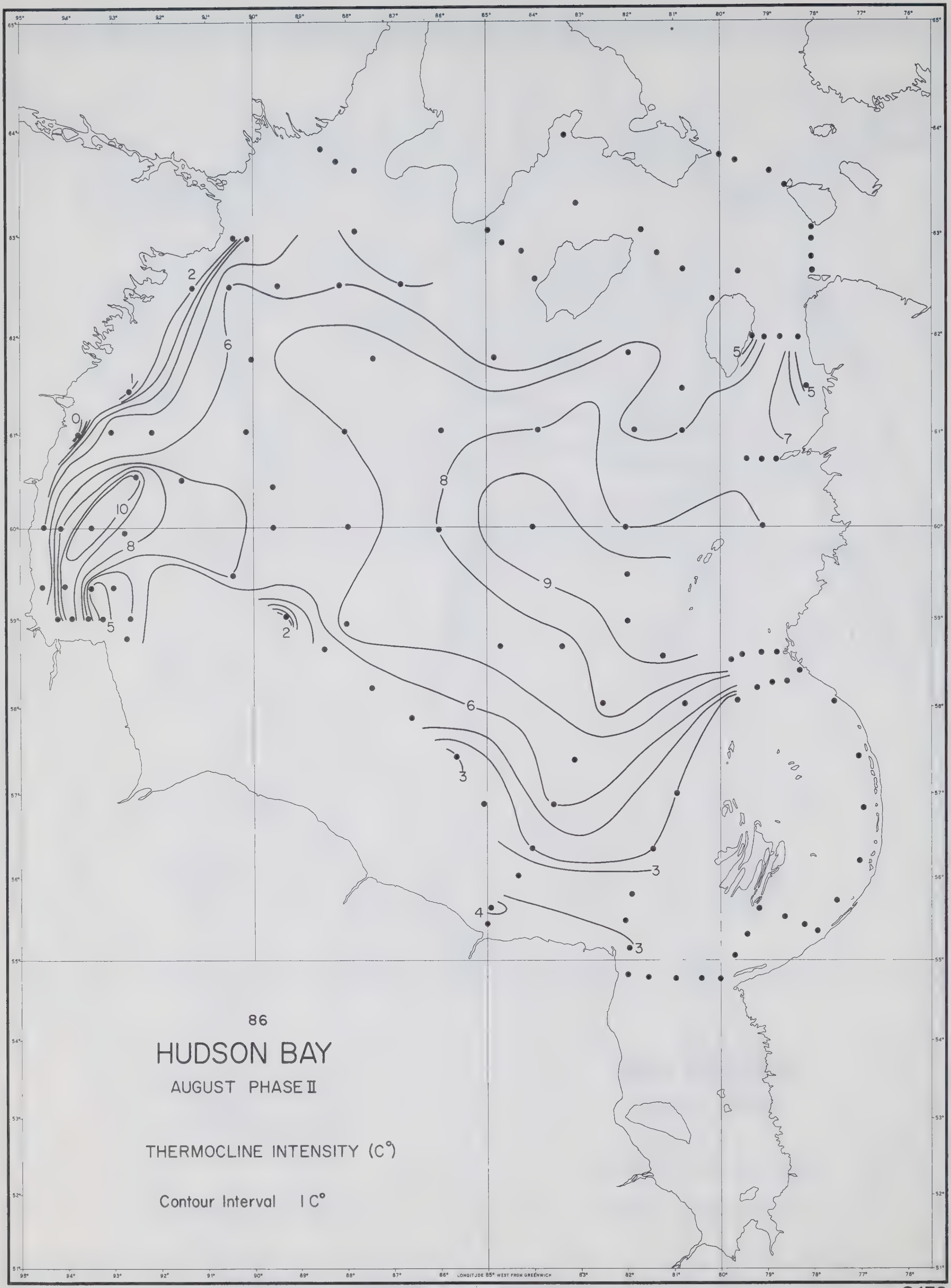
HUDSON BAY

AUGUST PHASE II

MIXED LAYER DEPTH (m)

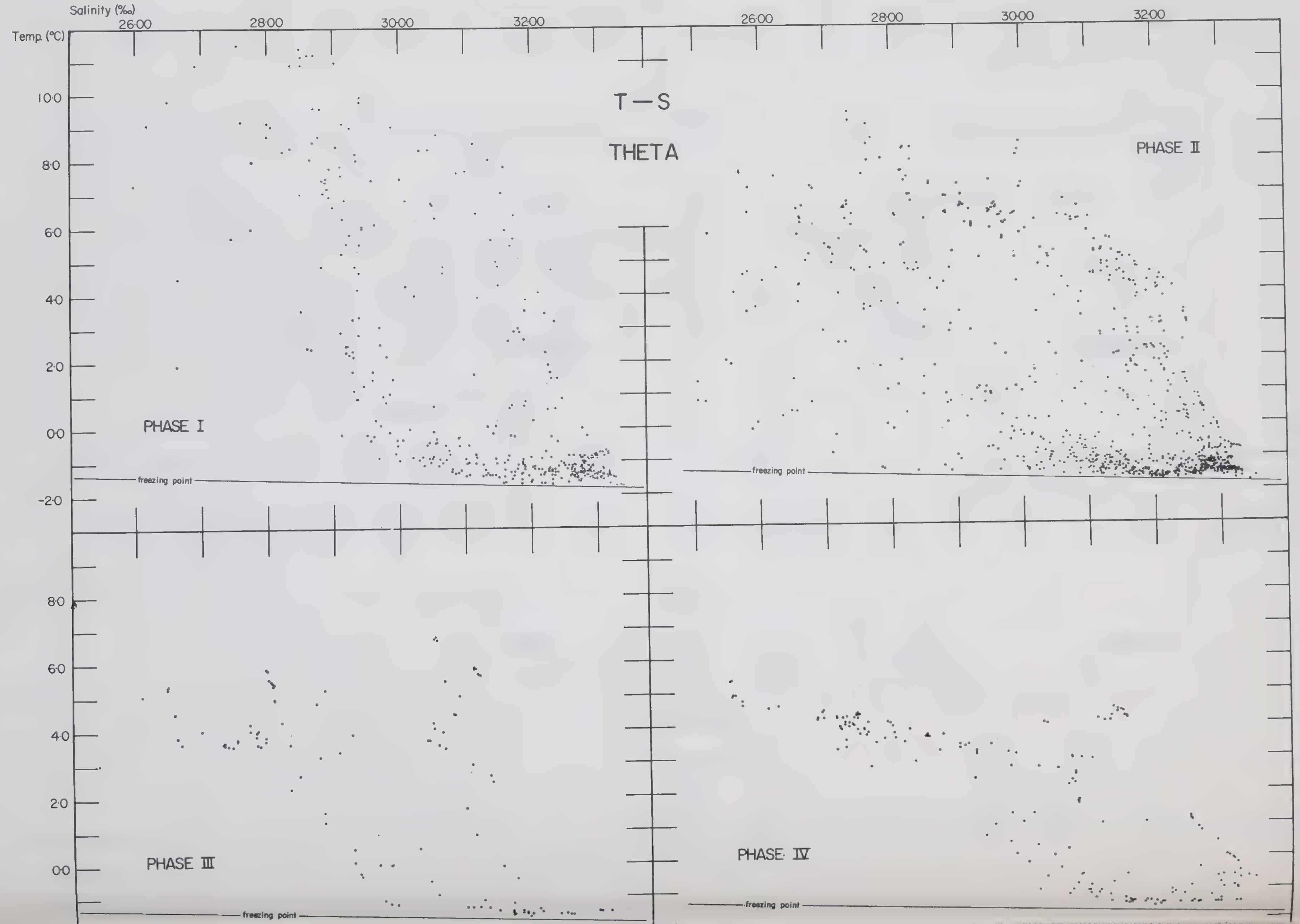
Contour Interval 5 m

Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

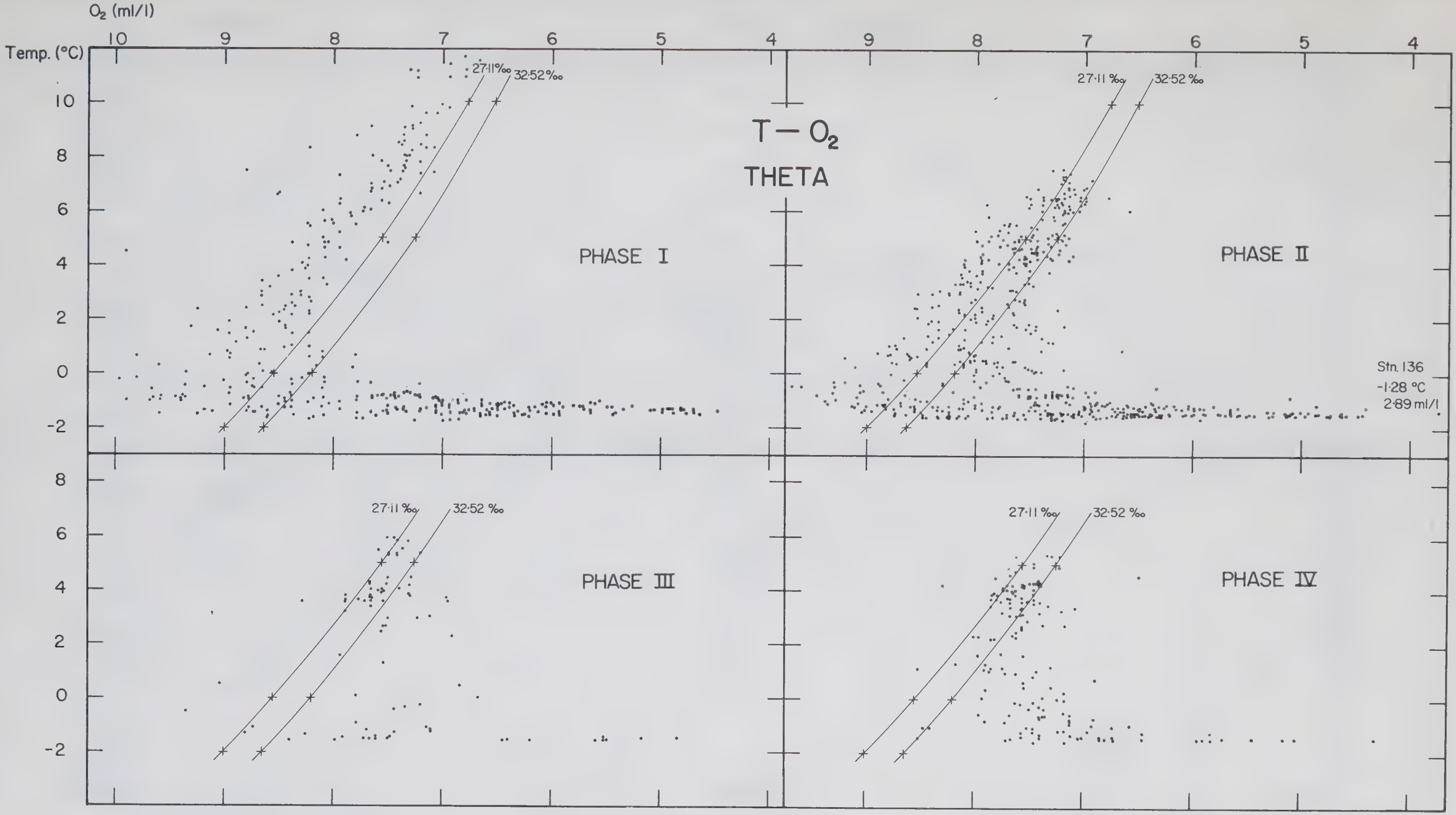


Published by the Canadian Hydrographic Service, Marine Sciences Branch,
DEPARTMENT OF MINES AND TECHNICAL SURVEYS, OTTAWA

S-176



Salinity (‰)
Temp. (°C)
26·00
28·00
30·00
32·00
10·0
8·0
6·0
4·0
2·0
0·0
-2·0
T—S
THETA
PHASE I
PHASE II
PHASE III
PHASE IV
freezing point



T — O_2
THETA
O_2 (ml/l)
Temp. (°C)
PHASE I
PHASE II
PHASE III
PHASE IV
27·11‰
32·52‰
Stn. 136
-1·28 °C
2·89 ml/l

MANUSCRIPT REPORT SERIES No. 2

The Tides in the Labrador Sea, Davis Strait and Baffin Bay

GABRIEL GODIN

Marine Sciences Branch
1966 Department of Mines and Technical Surveys, Ottawa

Manuscript Report Series No. 2

THE TIDES IN THE LABRADOR SEA, DAVIS STRAIT AND BAFFIN BAY

GABRIEL GODIN

1966

TABLE OF CONTENTS

THE TIDES IN THE LABRADOR SEA,

DAVIS STRAIT AND BAFFIN BAY

by Gabriel Godin

ABSTRACT

The tides in the sea made up of the Labrador Sea, Davis Strait and Baffin Bay, are investigated theoretically. It is shown that the tidal regime is controlled mainly by the oscillations of the Atlantic.

One dimensional calculations for M_2 and K_1, the main semidiurnal and diurnal constituents, yield profiles of the elevation along the sea which compare well with observations once proper account is taken of the diffuse reflection which takes place at the head. Currents of the order of 10 cm/sec are calculated for M_2 at the mouth of the Sea of Labrador, while the currents due to K_1 are insignificant everywhere in the sea except in Davis Strait. The maximum semidiurnal streams as well are found in Davis Strait where they are of the order of 15 cm/sec.

Two dimensional calculations allow us to take account of the earth's rotation. In order to carry these out, the sea is schematized into three basins of constant width and depth. With such a schematization it is possible to make some headway into the calculations once the concept of a narrow deep elongated sea is introduced. Such a sea is the intermediate between an infinite channel of constant depth and a broad rectangular sea of constant depth. As a consequence it will be shown that the motion in such a basin is controlled mainly by Kelvin waves and that the Poincaré waves are important only in the immediate vicinity of the boundaries. The motion is such a sea is described in a series of theorems. Once these are established it is possible to obtain a solution for M_2 for the model. A solution for K_1 fails to exist due to divergences at the inner boundaries.

Using the solution for M_2, cotidal charts for the elevation and the transport per unit width are drawn as well as a map of the elements of the current ellipses. The agreement between the cotidal chart for the elevation derived from coastal elevations and the one derived from the theoretical calculations is most satisfactory considering the crudeness of the model used.

It is established that the motion at the mouth of the Sea of Labrador is purely of the Kelvin type. Over most of the sea, the motion is completely described by the superposition of two standing Kelvin waves.

NOTATION

The quantities which have dimension are followed by their dimensions in brackets.

M: mass (grams)   L: length (centimetres)   T: time (seconds)

a	Half width of a rectangular basin (L)
a_j	Vector component (always indexed)
A	Arbitrary constant (L, LT^{-1})
$A_{\pm}$	Column vector of arbitrary constants (LT^{-1})
A_0	Arbitrary constant pertaining to a Kelvin wave (LT^{-1})
b	Width of a channel (L, LT^{-1})
$\underline{b}_j$	Basic vector
B	Arbitrary constant (L, LT^{-1})
B_0	Arbitrary constant pertaining to a Kelvin wave (LT^{-1})
c_j	Vector component
C	Arbitrary constant (LT^{-1})
D	Arbitrary constant (LT^{-1})
e	Base of the natural logarithms
f	Function
$\underline{f}$	Function denoted as a vector
F	Gravitational force (MLT^{-2})
g	Acceleration due to gravity (LT^{-2})
h	Depth (L)
i	Imaginary unit ($i^2=-1$)
I	Imaginary part of. Thus A^I: the imaginary part of A

j Any positive integer

j_k A positive integer belonging to a set k of positive integers

$j' \equiv \frac{j\pi}{2a} \quad (L^{-1})$

$J_r(x)$ Bessel function of the first kind of order r and argument x

k A positive integer being either 1 or 3

$k^2 \equiv \frac{\sigma^2 - 4\omega^2}{gh} \quad (L^{-2})$

$K \equiv \sigma/\sqrt{gh} \quad (L^{-1})$

$K_0' \equiv \sigma/\sqrt{gh} \quad (L^{-1})$

$K_0 \equiv \left(\sigma/\sqrt{gh}\right) \cdot \frac{2a}{\pi}$

$K_j \equiv \left\{ j^2 - \frac{\sigma^2 - 4\omega^2}{gh} \cdot \left(\frac{2a}{\pi}\right)^2 \right\}^{\frac{1}{2}}$

$l \equiv \frac{\pi L}{2a}$ Dimensionless equivalent of length

L 1) Length of a rectangular basin (L)
2) Latitude

m Odd positive integer

M Odd positive integer (distinct from m)

n Even positive integer

N Even positive integer (distinct from n)

P_n Vector component of $\cosh \lambda' x$ in the direction of the basic vector $\cos n' x$

$P_{j_k n_i}$ Vector component of $\cos j_k' x$ in the direction of the basic vector $\cos n_i' x$ (When $i \equiv k$ the subindices i and k are dropped)

q_m Vector component of $\sinh \lambda' x$ in the direction of the basic vector $\sin m' x$

$Q_{j_k m_i}$ Vector component of $\sin j_k' x$ in the direction of the basic vector $\sin m_i' x$

r Order of a Bessel function

R Real part of. Thus A^R : the real part of A

t Time (T)

u Velocity component along the x direction (LT^{-1})

v Velocity component along the y direction (LT^{-1})

$\underline{V}$ A vector

x Distance along one direction of the Cartesian axes (L); $\frac{2\sigma}{\alpha\sqrt{gh}}$

y Distance along a direction of the Cartesian axes perpendicular to x (L)

X A function dependant only on the x variable; or $e^{-\alpha y}$

Y A function dependant only on the y variable

Z Elevation above mean level

α Exponential increase in depth per unit length (L^{-1})

$\alpha \equiv \frac{2a}{\pi g}$ (T^2)

β Exponential increase in width per unit length (L^{-1})

$\Gamma_\pm$ Matrices of coefficients

δ_{ij} Kronecker delta symbol $\begin{cases} = 1 & i = j \\ = 0 & i \neq j \end{cases}$

$\delta_\pm$ Column vectors of constant terms

$\delta_\pm$ Column vectors of the coefficients of B_o

$\zeta_\pm$ Column vectors of the coefficients of (C_1, D_1)

θ_j Vector component of v at the mouth of the sea in the direction of the basic vectors $\binom{\sin m'x}{\cos n'x}$

$\lambda \equiv 2\omega/\sqrt{gh}$ (L^{-1})

$\lambda' \equiv \left(2\omega/\sqrt{gh}\right).\left(\frac{2a}{\pi}\right)$

Λ Wave length (L)

σ Angular speed (T^{-1})

Σ Summation sign

ω Angular speed of the earth's rotation (T^{-1})

$\Omega \equiv \frac{2\omega\sigma}{gh}\left(\frac{2a}{\pi}\right)^2$ A small quantity for a narrow deep elongated sea

$\sim$ Of the order of

$\simeq$ Approximately equal to

$\equiv$ Is defined by or is identical to

$\longrightarrow$ Goes into or tends towards

$(-)$ (-1)

Double subscripts are used in the latter part of the thesis. These are necessary to indicate to which set of indices a given index belongs. Thus A_{nk} denotes an arbitrary constant affected by an even index belonging to a set k.

Overscripts as well are used and are circled to distinguish them from exponents.

PRELIMINARIES

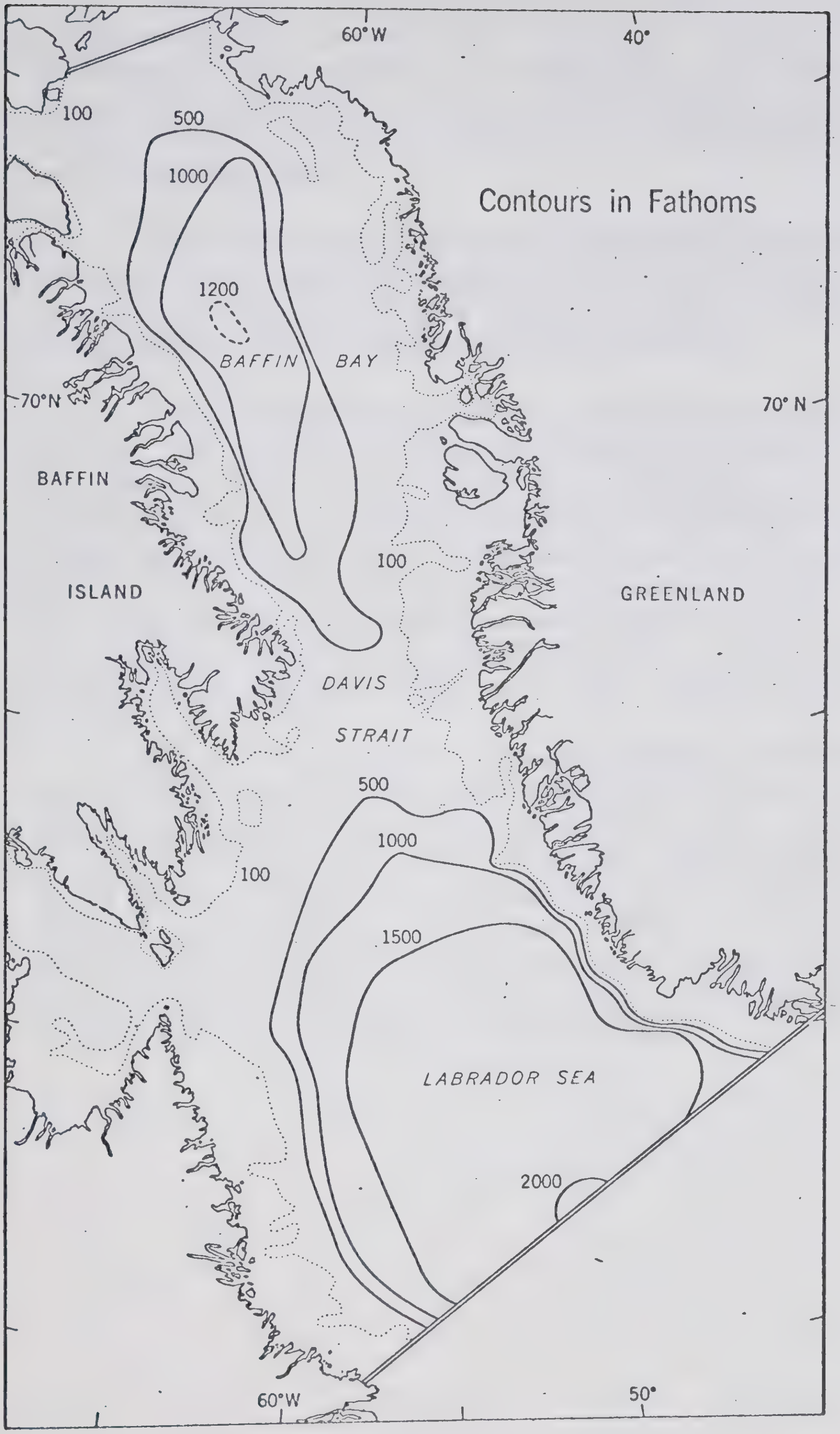

60° W
40°
Contours in Fathoms
100
500
1000
1200
BAFFIN BAY
70°N
70° N
BAFFIN
ISLAND
100
GREENLAND
DAVIS
STRAIT
500
1000
100
1500
LABRADOR SEA
2000
60°W
50°

Fig. 1

Physical and Tidal Characteristics of the Area Under Study

The Labrador Sea, Davis Strait and Baffin Bay are part of a single sea depicted in Fig. 1. The double lines indicate the boundaries chosen for this sea. The openings of Hudson Strait, Lancaster Sound, Jones Sound and Smith Sound denoted by the letters A B C D are narrow and shallow compared to the main body of the sea so that we may consider this one as approximately closed in its sides and northern boundary.

The Labrador Sea, Davis Strait and Baffin Bay form the three natural divisions of the domain under study; the first being broad and deep, the second narrow and shallow and the third being a bay insofar as it is connected to the outside ocean by a gully represented by Davis Strait.

The distance between the two boundaries shown in Fig. 1 is approximately 2350 km. The width varies between 970 km at the mouth to 300 km at the narrowest point in Davis Strait. If we transform the actual cross sections of the sea into rectangular sections of constant depth (the mean depth of the section), the depth of such sections varies from 2600 m at the mouth to 310 m at the shallowest point in Davis Strait located at 1140 km from the mouth. The greatest mean depth in Baffin Bay is 1100 m to be found at about 1800 km from the mouth. Fig. 4 shows the variation of width and mean depth along the sea along with an attempt of schematization will be discussed later. Fig. 4 is to be found on p. 25.

In spite of adverse climatic and geographic conditions, tidal information on this area has been slowly accumulating over the years and it is now possible to have a fair idea of the tidal regime prevailing there. Most of the observations however cover only short periods; the slow constituents therefore are not well known and our attention will be concentrated on the diurnal and semi diurnal tides. Tables 1 and 2 that follow give the data available: Table 1 gives the name and location of the station indicated by a number in Fig. 2 and 3 while Table 2 gives the pertaining tidal data in Atlantic Standard Time. Fig. 2 and 3 show cotidal charts for M_2 and K_1 based on such observations.

Table I
Tidal Stations

1.	Point Aldrich Cape Columbia, Grant Land	Lat. 83° 07'N Long. 69° 40'W 29 days 1908 U.S. Coast & Geod. Surv.
2.	Alert Ellesmere Island	Lat. 82° 30'N Long. 62° 20'W 2 x 29, 1 x 15 days 1960 C.H.S.
3.	Cape Sheridan Grant Land	Lat. 82° 27'N Long. 61° 30'W
4.	Fort Conger Discovery Harbour	Lat. 81° 44'N Long. 64° 44'W 369 days 1881 - 1882 U.S. Coast & Geod. Surv.
5.	Dundas Harbour Lancaster Sound	Lat. 74° 31'N Long. 82° 26'W 29 days Sept. 1960 C.H.S.
6.	Radstock Bay Devon Island	Lat. 72° 42'N Long. 91° 05'W 15 days Sept. 1961 C.H.S.
7.	Cape Hooper Baffin Island	Lat. 68° 21'N Long. 66° 45'W 30 days Aug. 1958 C.H.S.
8.	Kivitoo Baffin Island	Lat. 67° 56'N Long. 64° 56'W 30 days Aug. 1959 C.H.S.
9.	Broughton Island Davis Strait	Lat. 67° 31'N Long. 64° 04'W 15 days 1958 C.H.S.
10.	Kingua Fjord Cumberland Sound Davis Strait	Lat. 66° 30'N Long. 67° 20'W 41 days 1883 International Polarforschung
11.	Cape Dyer Baffin Island	Lat. 66° 33"N Long. 61° 39'W 30 days Sept. 1958 C.H.S.

12.	Brevoort Harbour Davis Strait	Lat. 63° 19'N Long. 64° 09'W 29 days 1957 C. H. S.
13.	Koojesse Inlet Frobisher Bay	Lat. 63° 43'N Long. 68° 41'W 30 days 1958 15 days 1959 C. H. S.
14.	Lewis Bay Frobisher Bay	Lat. 63° 20'N Long. 67° 57'W 29 days Oct. 1959
15.	Whiskukun Frobisher Bay	Lat. 63° 13'N Long. 68° 03'W 30 days Sept. 1958 C. H. S.
16.	Frobisher Farthest Frobisher Bay	Lat. 63° 29'N Long. 68° 02'W 29 days 1951 C. H. S.
17.	Tanner Bay Resolution Island	Lat. 61° 37'N Long. 64°44'W 2 days Sept. 1952 C. H. S.
18.	Port Burwell Hudson Strait	Lat. 60° 25'N Long. 64° 52'W 15 days 1885 Proc. Roy. Soc. London 45 1888-89
19.	Williams Harbour Labrador	Lat. 60° 00'N Long. 64° 19'W 15 days 1952 C. H. S.
20.	Nain Labrador	Lat. 56° 33'N Long. 61° 45'W 29 days 1932 Admiralty
21.	Hopedale Labrador	Lat. 55° 07'N Long. 60° 13'W 15 days 1959 C. H. S.
22.	Makkovik Labrador	Lat. 55° 06'N Long. 59° 10'W 30 days 1958 C. H. S.

23.	Cartwright Labrador	Lat. 53° 42'N Long. 57° 02'W 29 days 1934 Admiralty
24.	Cape Bryant Greenland	Lat. 82° 21'N Long. 54° 30'W 29 days 1909 U.S. Coast & Geod. Surv.
25	Thank God Harbour Polaris Bay, Greenland	Lat. 81° 36'N Long. 61° 40'W 174 days U.S. Coast & Geod. Surv.
26.	Rensselaer Greenland	Lat. 78° 37'N Long. 70° 53'W 116 days 1853-54 U.S. Coast & Geod. Surv.
27.	Port Foulke Greenland	Lat. 78° 18'N Long. 73° 00'W 58 days 1860-61 U.S. Coast & Geod. Surv.
28.	Thule Greenland	Lat. 76° 32'N Long. 68° 54'W
29.	Kamarajuk Fjord Umanak District	Lat. 71° 00'N Long. 51° 00'W 2 x 15 days 1930 Wegener
30.	Godhavn Disko Island, Greenland	Lat. 69° 15'N Long. 53° 33'W 29 days 1936 Danish Hydrographic Office
31.	Nunarssuaq Kronprinsens Ejland, Greenland	Lat. 68° 59'N Long. 53° 21'W 3 x 29 days 1950 Danish Hydrographic Office
32.	Egedesminde Greenland	Lat. 68° 43'N Long. 52° 53'W 29 days 1952 Danish Hydrographic Office
33.	Aningaq Rifkol, Greenland	Lat. 67° 55'N Long. 53° 50'W 3 x 29 days 1950 Danish Hydrographic Office
34.	Godthaab Greenland	Lat. 64° 11'N Long. 51° 45'W 145 days 1951 Danish Hydrographic Office

35. Faeringehavn
Greenland

Lat. 63° 42'N Long. 51° 33'W
3 x 15 days 1952
Danish Met. Office

36. Grønnedal
Greenland

Lat. 61° 13'N Long. 48° 07'W
29 days 1958
Danish Hydrographic Office

37. Julianehaab
Greenland

Lat. 60° 43'N Long. 46° 02'W
145 days 1934
J. Egedal
J. Thule Expedition

38. Nannortalik
Greenland

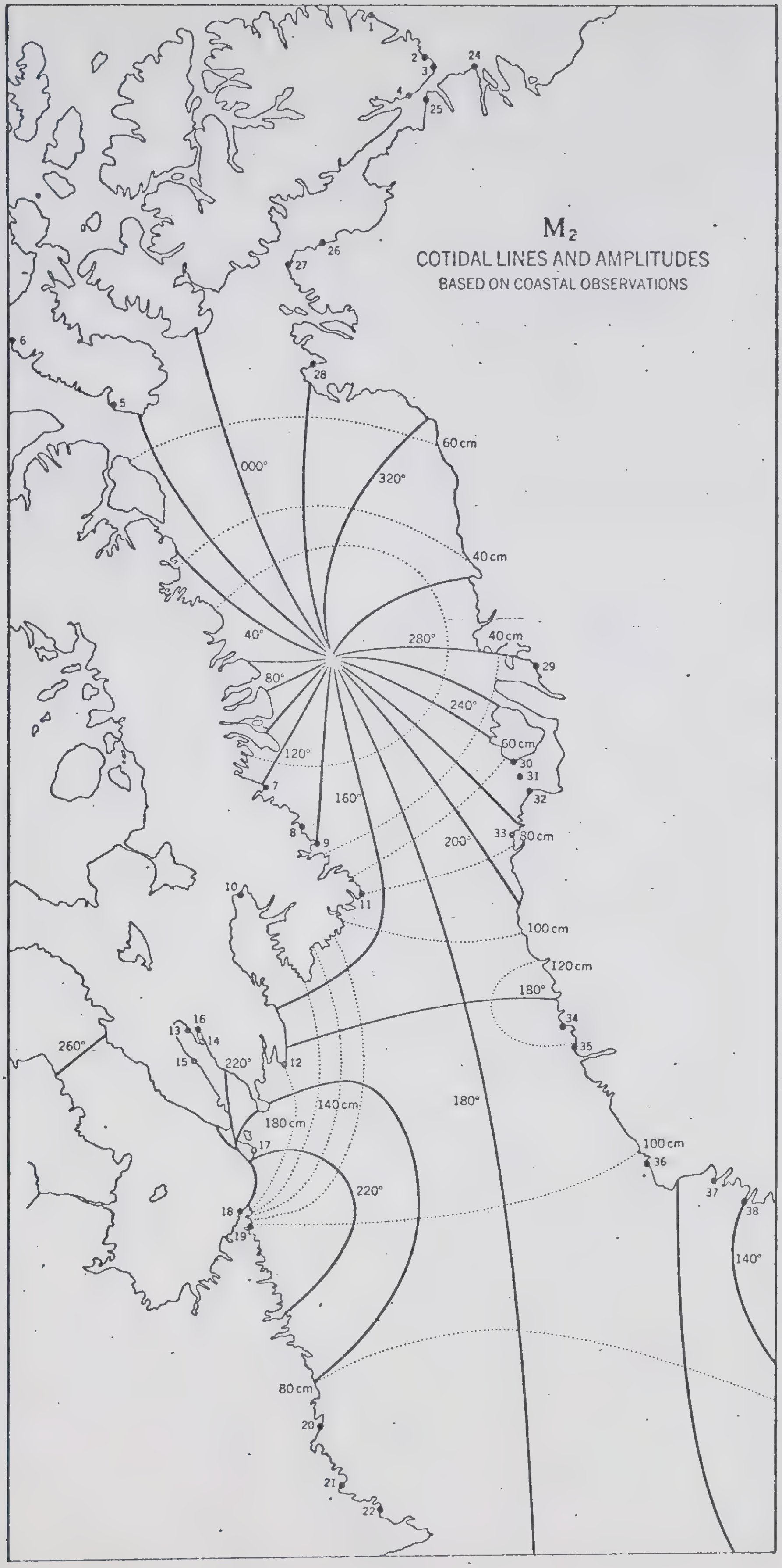
Lat. 60° 10'N. Long. 45° 20'W

Table II

ZONE +0400

STN	K1		O1		P1		M2		S2		N2	
	Z	G	Z	G	Z	G	Z	G	Z	G	Z	G
	CM	°	CM	°	CM	°	CM	°	CM	°	CM	°
1	5	321	3	295	2	321	12	254	5	304	2	222
2	5	287	3	261	2	287	22	278	11	321	3	342
3	5	299	3	285	2	297	25	310	12	354	4	285
4	9	226	3	208	2	238	60	348	27	028	12	325
5	34	203	11	170	11	208	66	022	26	060	14	348
6	27	243	17	205	9	243	63	013	26	049	11	001
7	19	191	13	142	6	191	20	122	6	180	3	089
8	21	189	6	145	7	189	247	138	11	186	5	088
9	25	189	8	123	8	189	31	138	13	180	5	116
10	8	039	3	050	3	039	227	178	81	217	37	144
11	11	196	3	132	4	096	84	157	25	199	13	117
12	17	064	10	029	6	064	181	196	64	225	35	161
13	18	123	10	063	6	123	334	221	115	270	61	192
14	22	126	10	055	7	126	294	228	110	277	72	201
15	20	111	9	063	6	111	334	226	119	272	67	201
16	18	101	9	062	6	098	321	219	110	267	54	189
17	32	101	11	070			175	214	58	257		
18	15	119	6	166	5	119	217	277	71	305	20	323
19	19	096	10	032	6	095	98	231	24	272	30	197
20	12	129	8	102	4	129	75	199	22	227	15	188
21	8	115	7	071	3	107	62	192	23	214	14	177
22	10	116	7	095	3	116	56	190	21	222	12	355
23	11	120	6	065	4	120	53	198	20	231	11	180
24	10	281	4	261	3	286	13	357	7	036	2	347

25	12 248	5 215	4 246	55 359	25 036	11 337
26	26 204	13 165	9 204	103 359	46 034	21 333
27	32 200	12 163	11 200	111 352	46 031	20 327
28	40 196	12 154		79 339	31 012	15 316
29	37 139	9 134	12 139	46 284	17 302	11 282
30	32 161	13 130	10 161	60 254	25 288	17 250
31	35 168	12 125	11 168	57 260	27 295	17 225
32	36 148	11 126	12 148	67 235	26 265	12 210
33	36 160	9 107	12 160	78 235	32 262	16 217
34	20 114	10 077	7 114	138 176	57 210	28 156
35	18 110	10 086	6 110	117 174	40 230	18 142
36	16 104	10 063	5 104	93 162	35 196	14 133
37	16 098	9 063	5 098	87 180	33 182	17 128
38	19 099	11 060		88 142	38 173	



M2
COTIDAL LINES AND AMPLITUDES
BASED ON COASTAL OBSERVATIONS
000°
320°
60 cm
40 cm
40°
280°
40 cm
80°
240°
60 cm
120°
160°
200°
80 cm
100 cm
120 cm
180°
260°
220°
140 cm
180 cm
180°
100 cm
220°
140°
80 cm

Fig. 2

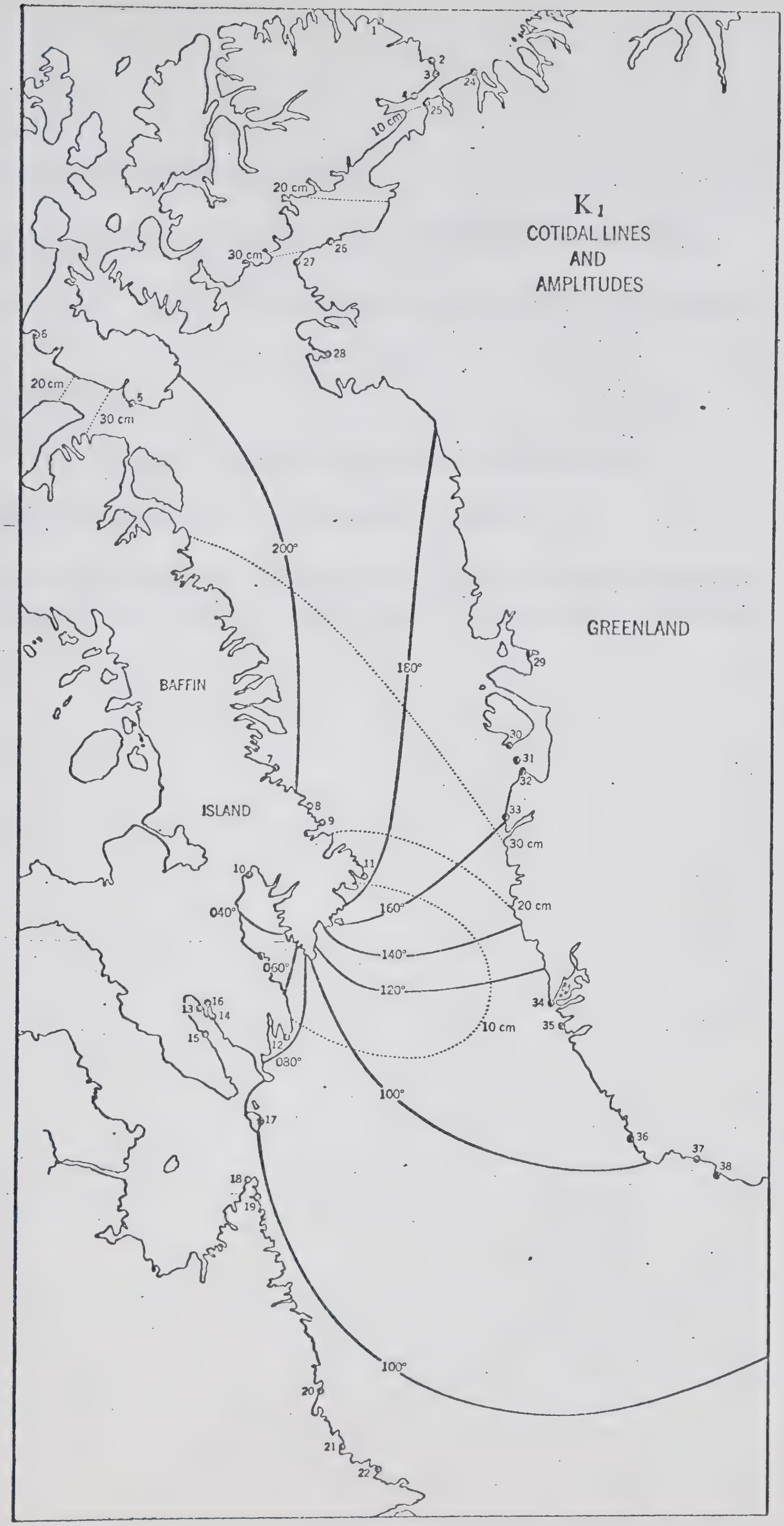

K1
COTIDAL LINES
AND
AMPLITUDES
GREENLAND
BAFFIN
ISLAND
10 cm
20 cm
30 cm
200°
180°
160°
140°
120°
100°
040°
060°
080°
100°

Fig. 3

The remarkable features of the M_2 tide are:

- The large amplitudes on the west coast in the vicinity of Hudson Strait,
- The presence of a point of amphidromy in the vicinity of latitude 72° 45'N.

Those of the K_1 tide are:

- The presence of a maximum in amplitude near the head of the sea,
- A degenerate point of amphidromy in the vicinity of 66° N.

N_2 , S_2 on one hand and O_1 on the other hand, exhibit the same characteristics; the position of their points of amphidromy varies slightly from that of M_2 and K_1 .

Review of the Literature

Harris (7) and Sterneck (14) gave some cursory comments about the tides prevailing in the area under study. The observational material was then very scanty and they had to limit themselves to generalities. Harris states that the tidal wave is of a standing character; the contraction of Davis Strait suggests to him that the cotidal lines there are crowded, which is not the case. Sterneck as well draws a set of cotidal lines across the sea.

Defant (2), in need of a value for the transport at the mouth of the sea of Labrador in his study of the tides of the Atlantic, applied his canal theory to this area which was eminently suitable for such an approach, and he obtained remarkably good results for M_2 . He located the node at the very location where subsequent observations would place it and he indicated that the maximum amplitude of $M_2$was to be found ahead of the contraction of Davis Strait which again conforms strictly to reality.

Recently, the two Russian workers Al'tschuler and Vladimirov (1) applied Hansen's (6) (so called Polukarov's) method for the calculation of the M_2 tide from the boundary values to the same area. This method represents one of the most significant advances in this field of knowledge; on the other hand its success depends on the quality of the boundary values. In this instance the Russian workers seem to have interpolated values for their stations #23 (70° 22'N, 67° 52'W) and #24 (71° 33'N, 71° 30'W) which lie on Baffinland from stations lying on Greenland. This had disastrous consequences since the two land masses lie across a point of amphidromy and there is a very marked change of phase from one side to the other. The results of their calculations reflect exactly the features they had introduced into their data, this is to say, a set of cotidal lines lying across the sea.

In our work we shall study the tidal motion in channels and rotating seas. The equations of tidal motion in channels without friction are simple and such a type of motion is adequately surveyed in Lamb (9) who describes in particular the work of McCowen (10) and Green (4). The form of the solution varies with the law assumed for the variation in width and depth.

Motion in rotating seas is analytically much more difficult and we restrict ourselves to rectangular seas of constant depth. In this case the equations can be solved with the help of eigenfunctions. Using the standard apparatus of vector spaces it is possible to investigate the motion in such a sea.

The study of tidal motion in rotating rectangular seas of constant depth was initiated by Thomson (16) and Poincaré (11). In 1921 Taylor (15) solved the problem of the reflection of a Kelvin wave in a semi infinite rectangular sea of constant depth; he showed that both the Kelvin and the Poincaré waves are necessary to give a complete solution and that only by assuming their simultaneous presence is it possible to satisfy any boundary condition. In 1931 Grace (3) introduced some simplification into the solution given by Taylor by assuming the smallness of $\left(\frac{\sigma\alpha}{gh}\right)^2$

nd he showed that the effect of the boundary value at the mouth could then be seen xplicitly in the solution.

Baffin Bay, Davis Strait and the Labrador Sea, when schematized as ectangular seas of constant depth, happen to satisfy very well Grace's assumption f the smallness of $\left(\frac{\sigma\alpha}{gh}\right)^2$ and this will lead to great simplifications.

General Summary

The diurnal tides and semidiurnal tides in the Labrador Sea, Davis Strait and Baffin Bay have been investigated with the help of the following assumptions:

1. – The sea under study is closed in its sides and northern boundaries.

2. – The wave motion is of a purely standing character with perfect reflection taking place at the head.

3. – The tide is induced exclusively by that in the Atlantic Ocean.

4. – The friction is negligible throughout the sea.

Initially we assume further that the motion is purely unidirectional. The equations are then very simple to solve and the form of the solution depends on the law of variation chosen for the width and depth. We chose an exponential law of width and depth when a schematization of constant depth and width was not obvious. By dividing the sea into an increasing number of channels it was possible to reproduce more and more closely the physical features of the sea. Fig. 4 illustrates the various steps in our schematization. It was then possible to obtain solutions for the elevation Z and the current v and the curves for Z are compared with the observed values. This is shown in Fig. 5 and Fig. 7. Even the more refined 6 channel model fails to locate the nodes accurately.

The assumption of perfect reflection at the northern boundary was then dropped and transports were assumed at this point while perfect reflection was assumed to take place at some undetermined point further beyond our chosen boundary for the sea. In this way it was possible to bring the node to its observed location. Fig. 6 and 8 illustrate the transports corresponding to the various Z curves; such transports imply estimates of the currents about which no information is available.

Once the position of the nodes and the transports at the head were established with the help of the one dimensional calculations, the problem of two dimensional motion was approached by schematizing the sea into three rectangular basins of constant depth. This took roughly into account the variation in width and depth while it was possible to obtain a solution of the equations of hydrodynamic which was not too difficult to handle. It was noticed that each basin satisfied very well the condition of narrowness and depth already noticed by Grace (3). The consequences of this were systematically investigated using as a powerful auxiliary the concept of linear vector space. In the body of the thesis we state and prove the following theorems:

<u>In a narrow deep and elongated sea,</u>

1. – The Poincaré waves are significant only in the vicinity of the boundaries

and are rapidly damped away from them.

A corollary is:
The motion tends to become unidirectional away from the boundaries.

2. - If motion is induced at one section, only the contribution to the Kelvin waves will propagate far into the sea, the contribution to the Poincaré waves will rapidly be damped.

We have two corollaries:

a. Only the part of the energy which is transmitted to the Kelvin wave will propagate into the sea.

b. Only the part of the current which is constant across the mouth will propagate into the sea.

3. - A Kelvin wave can always be reflected in a rotating sea of any length.

Once these theorems are established it is a simple matter to state and satisfy the boundary conditions. The calculations are lengthy and tedious but straightforward. We obtain a solution for M_2 which converges everywhere except at the inner corners while the solution for K_1 diverges at the inner boundaries. The results of such calculations are shown in Fig. 9 to 12.

Basic Approximations

We wish to assume first that the sea under study is approximately closed in its sides and in its northern boundary.

The main opening on the sides is Hudson Strait. The cross section at the mouth of Hudson Strait is 20 km^2 while the section across the sea just north of the opening of the strait (around 62° 30'N) is 630 km^2 . The ratio of the two cross sections is of 1 to 31.

Baffin Bay ends into the three sounds of Lancaster, Jones and Smith. We assume that the bay is closed at the sect ion marked by the double lines shown in Fig. 1 around latitude 75° N. The largest section of Baffin Bay (around 72° 30'N) is 430 km^2 while those of Lancaster and Smith Sounds are 73 km^2 and 71 km^2 respectively. The section of Jones Sound is negligible. The assumption of complete closure at the northern end is therefore not too strong and will have to be relaxed eventually.

Friction is not taken into consideration. The maximum tidal streams over the body of the sea (except in the immediate vicinity of the head) are computed to be of the order of 15 cms/sec and the minimum mean depth is 300 m. In such a situation the effect of friction is of secondary importance.

The tides are considered as being exclusively induced by those in the Atlantic; the tides caused in situ by the tidal potential are neglected. The following calculations will give and idea of the magnitudes involved.

The force due to M_2 and K_1 is given by

$$F = g\,(.7607\times10^{-7} \quad \cos^2 L, \quad .4442\times10^{-7} \quad \sin 2L)$$

where L is the latitude and where the time varying part has been removed (see Schureman (13)).

The sea under consideration can be roughly considered as a rectangular basin of constant depth 1120m and of length 2350 km lying between latitude 55° and 75°N (.96 to 1. 31 radians) at an angle to the meridians; we wont specify this angle but it will be such that L is approximately given by

$$L = 1.31 - fy$$

where y is the distance measured from the head and f has for value 1.49×10^{-4} km^{-1} so that L = . 96 radians when y = 2350 kms.

Deriving an equation for Z from equations (1-1) and (1-2) in Appendix 1 with the added force terms we obtain

$$\frac{d^2 Z}{dy} + K^2 Z = (.7607\times10^{-7} \quad f \sin 2L, \quad .8884\times10^{-7} \quad \cos 2L) \text{ for}$$

M_2 and K_1

This is a non homogeneous linear differential equation with constant coefficients which has for solution

$$Z = A \cos Ky + B \sin Ky + \left(.7607\text{x}10^{-7} \frac{f \sin 2L}{k^2 - 4f^2}, -.8834\text{x}10^{-7} \frac{f \cos 2L}{k^2 - 4f^2}\right)$$

or explicitely

$$Z = A \cos Ky + B \sin Ky + (.66 \sin 2L, -3.4 \cos 2L)$$

where A and B are arbitrary constants of integration. If we impose as boundary conditions $Z = 0$, $v = 0$ at $y = 0$, the solution becomes

$$Z = (-.32 \cos Ky - .13 \sin Ky + .66 \sin 2L, -2.97 \cos Ky -.37 \sin Ky - 3.4 \cos 2L)$$

The largest values of Z, which obtain at the mouth, are .93 cms and .79 cm for M_2 and K_1 respectively. This type of motion can then be safely neglected.

ONE DIMENSIONAL THEORY

The Equation of Motion in One Dimension for a Fluid and the Boundary Conditions: The Solutions

The equations of motion for a fluid of density 1 in one dimension are:

$$\frac{\partial v}{\partial t} = -g \frac{\partial Z}{\partial y} \qquad (1)$$

$$\frac{1}{b} \frac{\partial}{\partial y} (b h v) = -\frac{\partial Z}{\partial t} \qquad (2)$$

These hold for a channel of rectangular section, width b, depth h. (1) states Newton's second law while (2) states the law of conservation of mass. The conditions of validity of these equations are well known and will not be reproduced here.

Z and v are periodic functions of time in the phenomena under study and we write, assuming standing waves,

$$Z(y, t) = Z(y) \quad \cos \sigma t$$

$$v(y, t) = v(Y) \quad \sin \sigma t$$

where σ represents the rate of change of the phase of the given constituent and t is the time.

This notation leads to no ambiguity since from now on we shall be concerned only with the space part of Z and v. The equations satisfied by each of these functions are now

$$v = -\frac{g}{\sigma} \frac{dZ}{dy} \qquad (3)$$

$$\frac{1}{b} \frac{d}{dy} (b h v) = \sigma Z \qquad (4)$$

Our models will consist of one or more channels open at one or both ends or closed at one end. We therefore wish to solve (3) and (4) for the following boundary conditions.

$$Z = 1 \text{ cm at the mouth}$$

$$v = 0 \text{ cm/sec at the head.}$$

We impose the boundary value unity on Z because (3) and (4) are linear equations and one needs simply to multiply the solutions obtained with such a boundary value by Z_0 if this happens to be the value observed at the mouth.

Since our models may consist of more than one channel we can determine the arbitrary constants completely only if we impose further conditions relating the elevations and the transports at the junctions of every two channels. The standard connecting conditions holding at the junction of two channels, say 1 and 2 are

$$Z_1 = Z_2 \qquad (5)$$

$$b_1 h_1 v_1 = b_2 h_2 v_2 \qquad (6)$$

(5) excludes any discontinuity in amplitude while (6) even if it implies a possible jump in v conserves the mass of liquid flowing per unit time from one channel to another.

The solution of (3) and (4) depends on the functional dependence of b and h on y.

The solution comes out in terms of

1. - Trigonometric functions, if b and h are constant,

2. - Bessel functions, if b and h vary exponentially,

3. - Confluent hypergeometric functions, if b varies exponentially while h varies linearly,

4. - Hypergeometric functions, if b is constant while the depth is described by a parabola,

5. - Infinite series not related to any standard functions, when b and h vary linearly (These series represent Bessel functions of order 0, 1 and 2 when the width is constant or when the depth and width vary at the same rate).

Preliminary calculations have been done using these various laws when they were approximately equivalent to describe parts of the b h curve given in Fig. 4; the results have been found to be almost identical. In the final calculations presented here an exponential law of width and depth has been used throughout except when a schematization of constant width and depth was obvious. The general solution of (3) and (4) for an exponential law of width and depth comes out in terms of Bessel functions of the first kind and usually of a non integral order; it is derived in Appendix 1.

Although these functions are not tabulated and have to be evaluated from their series expansions, the convergence is rapid, which is not the case for the functions corresponding to the other laws of width and depth. More, extent tables

of Bessel functions of an integral order helped to check the calculations done using the Bessel functions of non integral order while it was more difficult to check the other calculations using the solutions involving the more esoteric functions which are only covered by rudimentary tables.

To set up the coastal observations on amplitude on a linear scale we drew a line in the middle of the sea following the trend of the coast. Distance was measured along this line and divisions approximately perpendicular to it were drawn to reach the coasts. In this way the amplitudes were plotted as in Fig. 5 or 7, the full points denoting the observations on the East Coast while the open circles give those on the West Coast. Points considered to lie beyond the node were plotted on the negative axis.

The non schematized plot of the width and depth variation along the sea shown for instance in Fig. 4 gives a three dimensional picture of the basin. One imagines the b plot to be reflected across the y axis (and reduced by one half) and this gives a top view of the basin, while the h plot indicates the variation of depth along it.

Table 3 that follows give the data on the schematization of this basin into 1, 2, 3 and 6 channels while Fig. 4 illustrates this schematization except in the one channel case.

The schematization into two channels represents roughly the Labrador Sea connecting into Baffin Bay, Davis Strait being done away with altogether; the minimum of depth and width are put together which is not the case in the actual sea. The three channel model includes Davis Strait while the six channel model molds itself reasonably well to all parts of the basin.

Figs. 5 to 8 show the results of the calculations carried through for M_2 and K_1. Appendix 2 gives the corresponding analytic solutions. These solutions hold for the boundary value $Z_0=1$ at the mouth. To obtain the curves shown in Figs. 5 and 7 the Z solution has been multiplied by 70 and 14 respectively for M_2 and K_1 since these seem reasonable boundary values; the transports however correspond to the normalized solutions. The latter part of Appendix 2 gives the currents corresponding to the six channel model since in that instance they are continuous.

Table III

DATA ON THE SCHEMATIZATION INTO CHANNELS

NO OF CHANNELS		LENGTH KM	WIDTH KM	$\beta x l$	DEPTH M	$\alpha x l$	β / α
1		2350			1120		
2	1	1140	970 → 300	-1.17351	2600 → 310	-2.12346	.5526
	2	1210	450		800		
3	1	1000	970 → 420	.83725	2600 → 310	2.12704	.3936
	2	140	420 → 300	- .33647	310 → 400	.24822	-1.3555
	3	1210	450		800		
6	1	600	970 → 800	- .19268	2600 → 1600	- .48551	.3969
	2	400	800 → 420	- .64434	1600 → 310	-1.63797	.3934
	3	140	420 → 300	- .33647	310 → 400	.24822	- 1.3555
	4	260	300 → 480	.47000	400 → 885	.79412	.5918
	5	800	480		885		
	6	150	480 → 290	- .50410	685 → 250	-1.26413	.3987

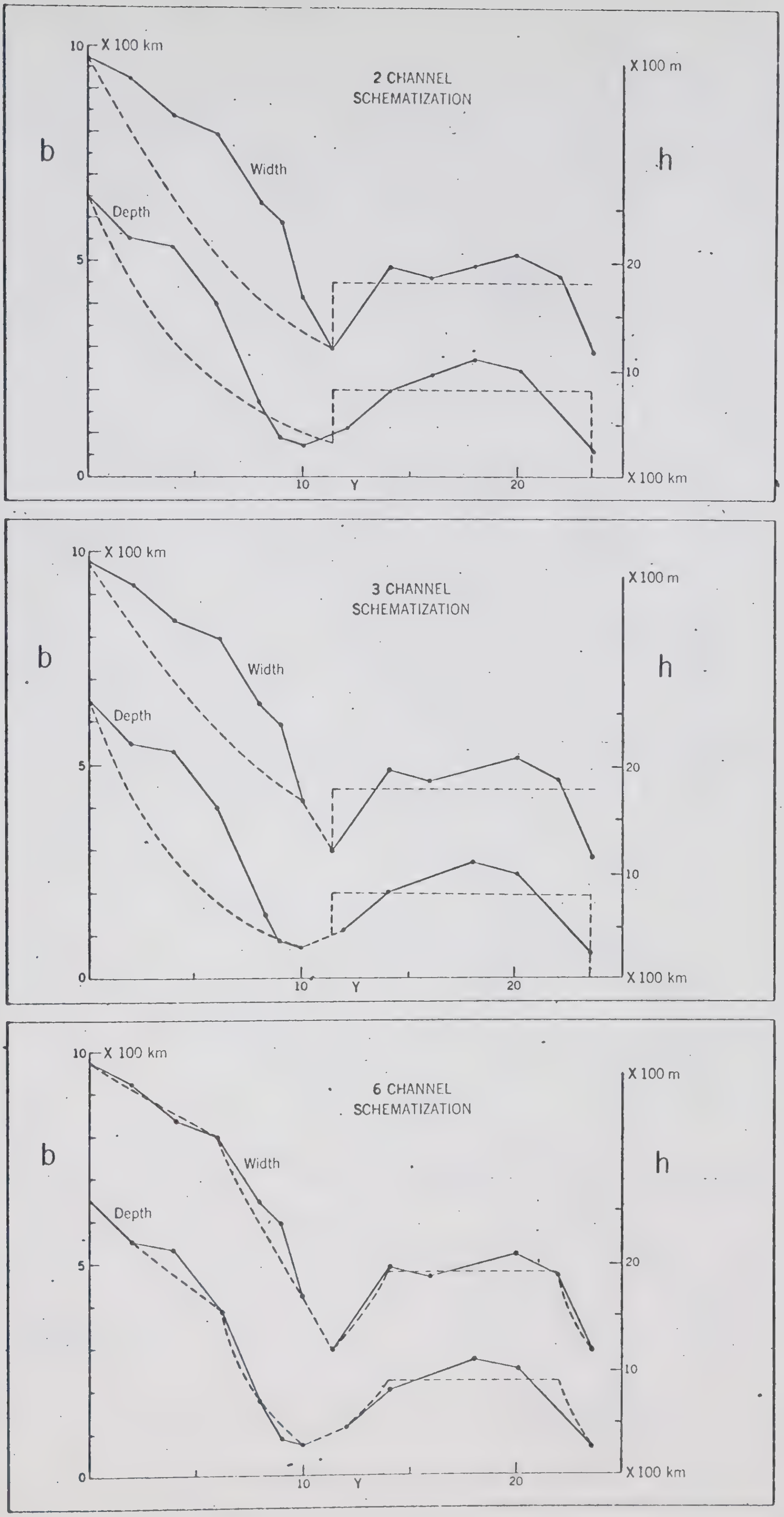

10
X 100 km
2 CHANNEL
SCHEMATIZATION
X 100 m
b
Width
h
Depth
5
20
10
0
10
Y
20
X 100 km
10
X 100 km
3 CHANNEL
SCHEMATIZATION
X 100 m
b
Width
h
Depth
5
20
10
0
10
Y
20
X 100 km
10
X 100 km
6 CHANNEL
SCHEMATIZATION
X 100 m
b
Width
h
Depth
5
20
10
0
10
Y
20
X 100 km

Fig. 4

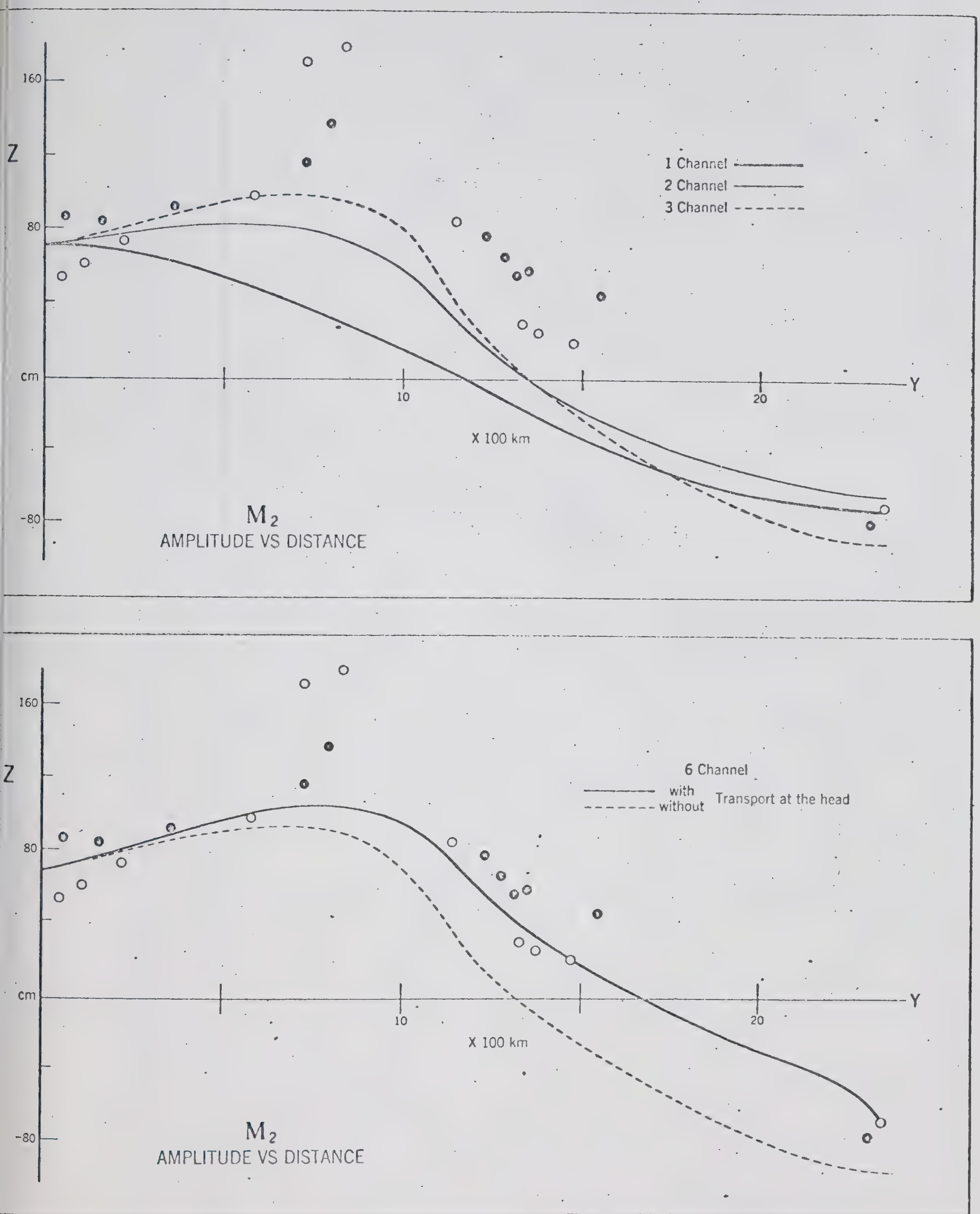
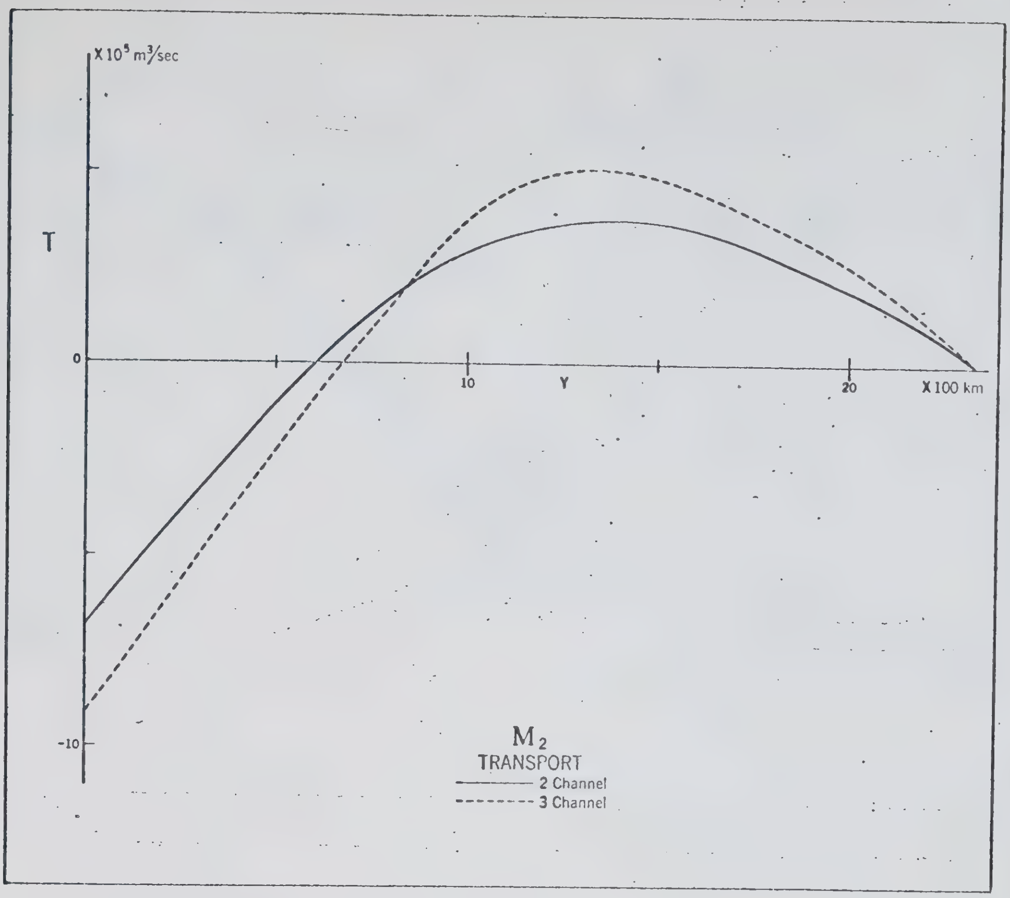

160
Z
80
cm
-80
10
20
Y
X 100 km
1 Channel
2 Channel
3 Channel
M2
AMPLITUDE VS DISTANCE
6 Channel
with
without
Transport at the head

Fig. 5


X 10⁵ m³/sec
T
0
10
Y
20
X 100 km
-10
M2
TRANSPORT
2 Channel
3 Channel

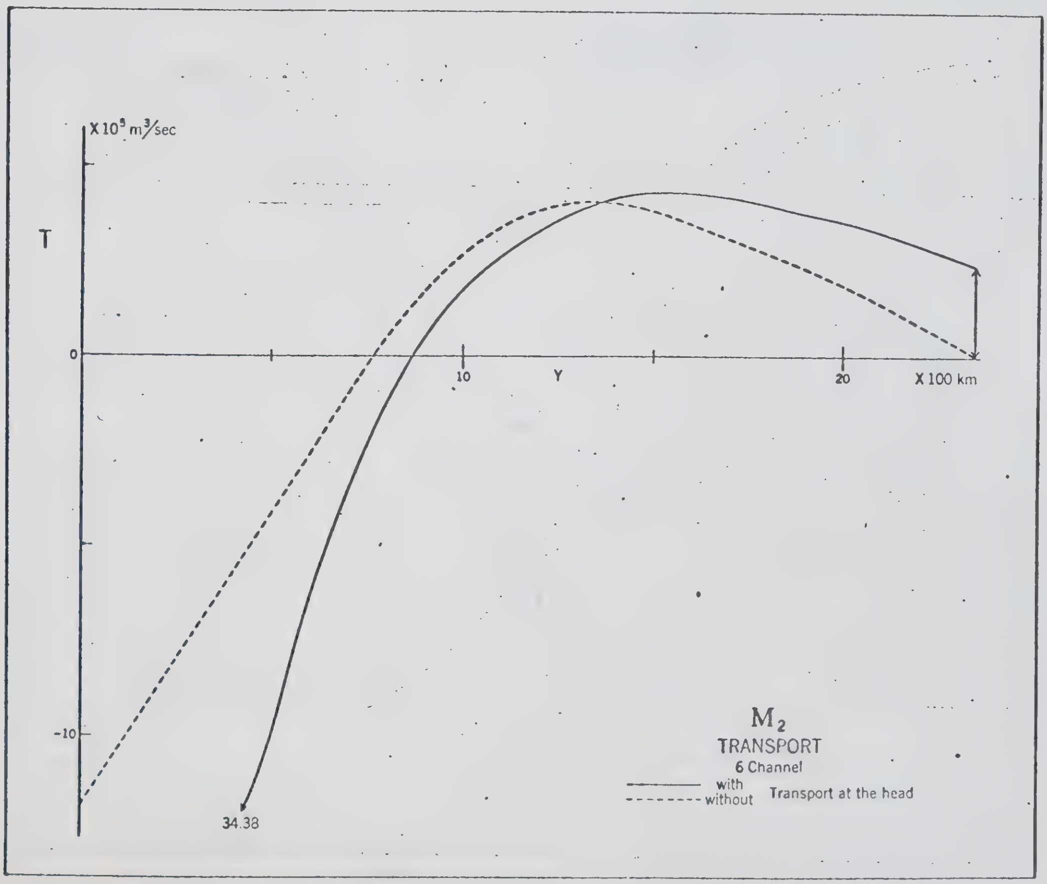
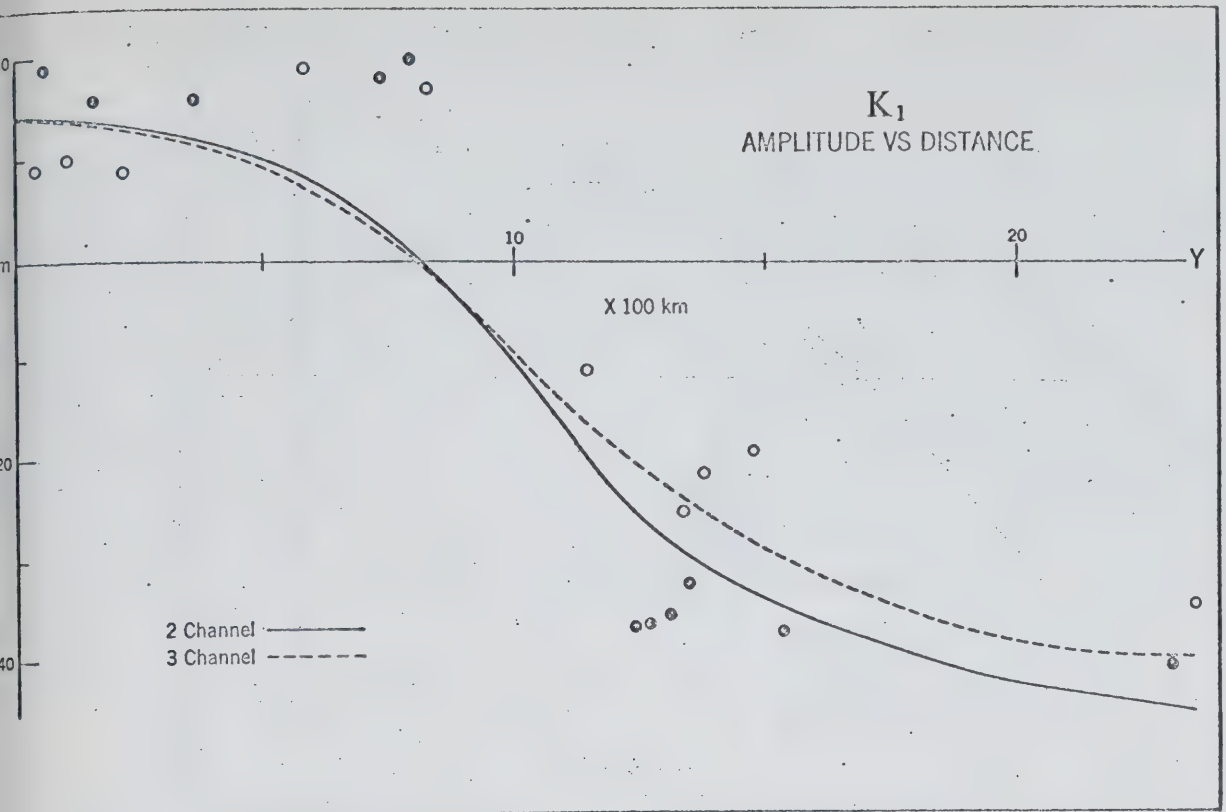

X 10⁵ m³/sec
T
0
10
Y
20
X 100 km
-10
34.38
M2
TRANSPORT
6 Channel
with
without
Transport at the head

Fig. 6


K1
AMPLITUDE VS DISTANCE
10
20
Y
X 100 km
20
40
2 Channel
3 Channel

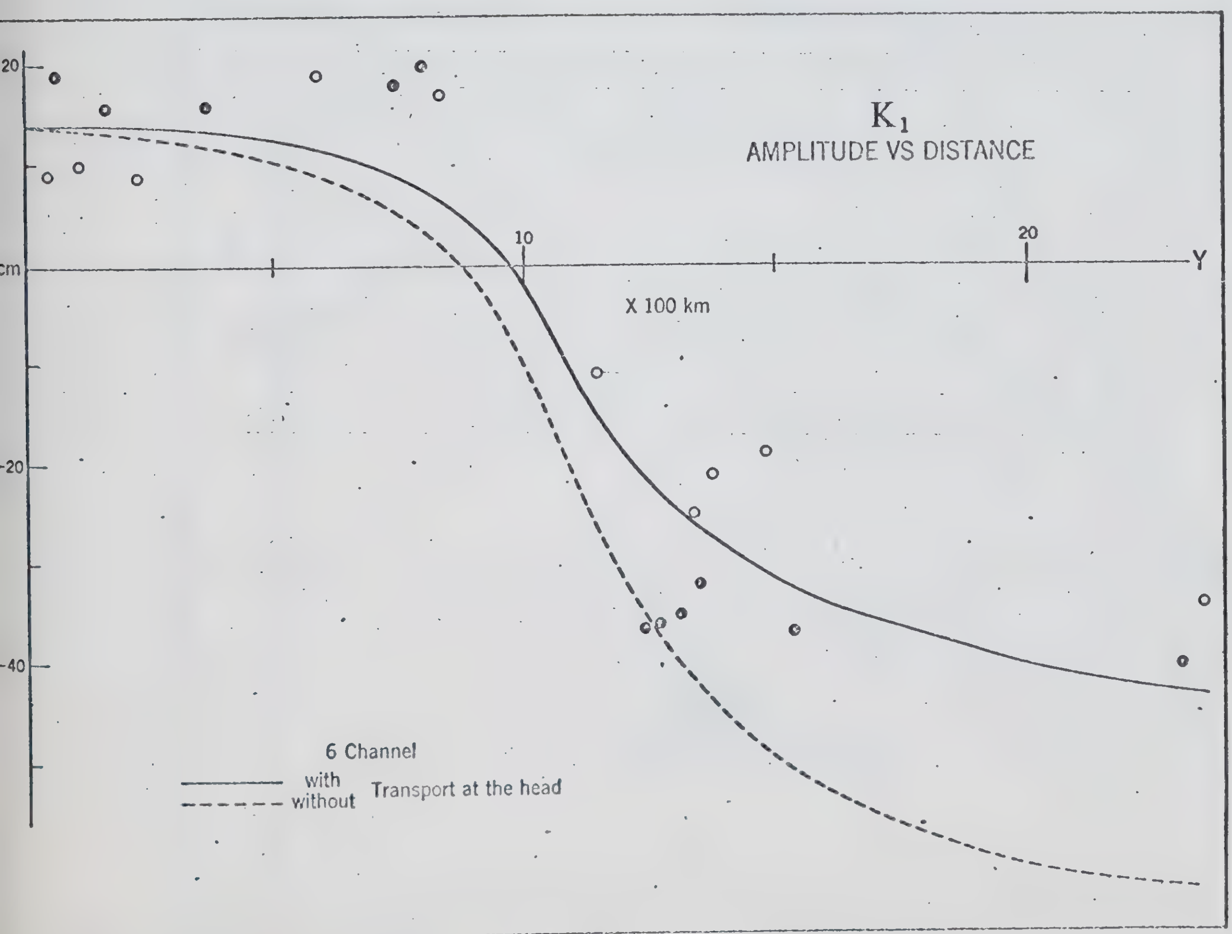

K1
AMPLITUDE VS DISTANCE
20
cm
10
20
Y
X 100 km
-20
-40
6 Channel
with
without
Transport at the head

Fig. 7

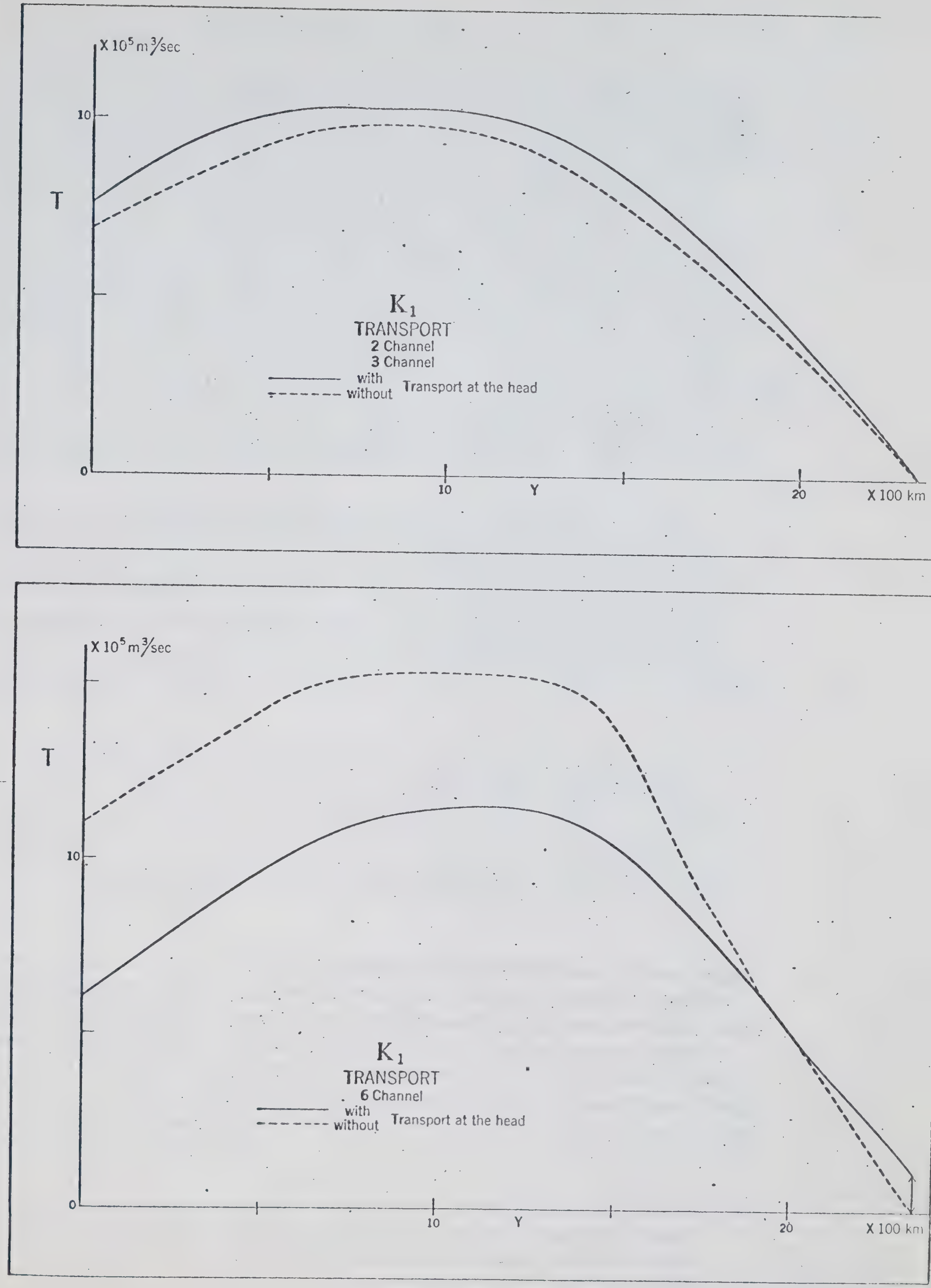

X 10^5 m³/sec
10
T
K_1
TRANSPORT
2 Channel
3 Channel
with
without
Transport at the head
0
10
Y
20
X 100 km
X 10^5 m³/sec
T
10
K_1
TRANSPORT
6 Channel
with
without
Transport at the head
0
10
Y
20
X 100 km

Fig. 8

Discussion of the Results

In Fig. 5 the 1 channel approximation for M_2 although very rough indicates the presence of a node half way up the channel; it fails however to show any amplification of the amplitude and the node is too close to the mouth. The 2 and 3 channel models show some amplification but still fail to bring the node forward enough. The 6 channel model, assuming perfect reflection gives quite similar results. Assuming a transport of 1.75×10^7 m^3/sec (for $Z_o = 70$ cm) at the head brings the node close to where it most likely lies. Some increase in amplification and a forward shift in the maximum is noted but such an amplification is far short of that which actually occurs on the west side of the sea. The amplification on the Greenland side is fully accounted for since the calculated amplification is of the order of 150% which is that observed on the Greenland coast. At the head of the sea the curve shows some steepening which indicates that there is little change in amplitude in the northern part of Baffin Bay till one reaches Lancaster Sound and Smith Sound. There is no data to support or contradict this deduction. The currents at the head are computed to be of the order of 24 cm/sec for a Z_o value of 70 cm.

In Fig. 7 the one channel approximation for K_1 is not shown. In this approximation, the node lies just inside the sea; this would indicate resonance, which is not the case. The 2 and 3 channel models give quite a reasonable fit to the amplitude assuming perfect reflection. The 6 channel model using the same assumption apparently leads to worse results. Assuming a slight outflow at the head of 1.7×10^6 m^3/sec, being equivalent to a stream of 2 cm/sec across the section, brings the curve closer to the observed values. This outflow corresponds to a boundary value Z_o of 14 cm.

If one inspects Table 1, one notes that in Lancaster Sound the amplitude of M_2 and K_1 at Dundas Harbour (Station 5) is larger than that at Radstock Bay (Station 6). There must be a maximum in the amplitude either between stations 5 and 6 or ahead of Dundas Harbour. Assuming this last instauce to be the case and the wave in this area to be of a perfect standing character, there should be weak streams in the vicinity of Dundas Harbour and the assumption of the closure of the sea in this area should hold well.

When we come to Smith Sound the situation is quite different. Thule (Station 28) shows the largest amplitude in K_1 and the diurnal streams should be weak in this area thus allowing us to assume perfect reflection of K_1 at this boundary. But the maximum in the amplitude of M_2 is to be found in the neighbourhood of Port Foulke (Station 27) and Rensselaer (Station 26) further up in Smith Sound. There are large gradients in the amplitude, from 66 cm at Thule to 111 cm at Port Foulke and therefore appreciable semidiurnal currents should be expected to prevail at the chosen northern boundary of the sea.

One notices the large increase in transport at the mouth for M_2 for a slight increase in outflow at the head. Mathematically this can be explained by the steepening of the gradient at the mouth for increasing outflows. The currents

at the mouth are larger and these flowing across a very large section will cause much increased transports. On the other hand such large transports are physically possible at the mouth of the sea since there exist very large gradients in the amplitude of M_2 along the Labrador Coast and even through these gradients do not extend all the way across the sea to Greenland, they must extend up to a certain distance from the west coast and cause very marked transports.

Comparison with Defant's Calculations

Defant (2) investigated the M_2 tide in the area covered by this thesis.

His data on width and depth are almost identical to those presented in Fig. 4 for the Labrador Sea and Davis Strait. However the depths shown for Baffin Bay are consistently lower than those that can be abstracted from contemporary bathymetric charts.

Using a numerical scheme for the integration of equations (3) and (4), M_2 was calculated for a 10 channel model assuming complete reflection at the head. The results indicate a node at $y = 1550$ km from the mouth (our calculations with the same assumption would locate it at 1350 km and a maximum in amplitude at about $y = 830$ km); this corresponds very closely to reality.

The success of these rather crude calculations using a defective model is surprising at first. However it must be realized that shallower depths in Baffin Bay automatically bring the node forward towards the head.

It was decided to repeat Defant's calculations using his data on width and depth; first for a 20 channel model of the sea using his numerical scheme, then for a 3 channel model consistent with our approach and defined by the following quantities

$l_1 = 950$ km	$b_1 = 969 \rightarrow 365$ km	$h_1 = 2610 \rightarrow 350$ m
$l_2 = 240$ km	$b_2 = 365$ km	$h_2 = 350$ m
$l_3 = 1190$ km	$b_3 = 471$ km	$h_3 = 610$ m

The results are given in Table 4. Displacements rather than currents are shown. The amplitudes are normalized to 100 at the mouth and such data on amplitude that were derived from the numerical schemes were interpolated to yield values at the mouth and head of the channels; the ξ's for the 3 channel model are omitted since they are discontinuous.

It can be noticed that the finer 20 channel scheme which should give a solution closer to the exact solution of (3) and (4) for the sea described by Defant modifies his solutions in two ways:

1) The maximum is reduced from 151 to 138,
2) The node is pushed away from the head and lies at about 1450 (the maximum as well being brought to lie at 700 km).

The 3 channel model although itself quite coarse tends to agree with the 20 channel model.

Table IV

Defant's Calculations Repeated

	AMPLITUDE			DISPLACEMENT	
	10 CHANNEL	20 CHANNEL	3 CHANNEL	10 CHANNEL	20 CHANNEL
CHANNEL JUNCTION	Z CM	Z CM	Z CM	ξ M	ξ M
0	100	100	100	387	371
0.5		109			371
1	119	117	115	374	362
1.5		124			285
2	136	130	125	298	238
2.5		134			215
3	151	138	127	257	159
3.5		136			024
4	151	124	120	- 260	- 471
4.5		97			- 1067
5	95	60	56	- 1910	- 1911
5.5		26			- 1364
6	20	0	15	- 1043	- 1038
6.5		-23	0		- 963
7	-30	-43	-28	- 878	- 838
7.5		-59			- 694
8	-66	-72	-63	- 558	- 533
8.5		-81			- 376
9	-86	-87	-88	- 224	- 216
9.5	(-92)	-90			- 156
10			-96	0	0

Conclusions

The one dimensional theory confirms the presence of nodes in the amplitudes of M_2 and K_1 in the area under study. It fails however to locate these nodes accurately unless some outflow is assumed at the head of the sea; these outflows correspond to currents of 24 cm/sec and 2 cm/sec for M_2 and K_1 respectively.

It supports rather well the variation in the amplitude of M_2 and K_1 along the sea; large amplitudes of the diurnal wave are calculated in Baffin Bay in spite of its smallness at the junction with the Atlantic and some amplification of M_2 ahead of Davis Strait is calculated.

The theory also has the advantage of supplying estimates of the currents prevailing in the sea and about which no information is available. It gives a mean current of 10 cm/sec for M_2 at the mouth of the basin and it suggests that the diurnal currents are small over most of the sea except in Davis Strait where they reach a magnitude of 13 cm/sec.

The one dimensional theory obviously cannot explain why the point of amphidromy for K_1 is degenerate, it cannot take account of the large gradients in the amplitude of both constituents across the mouth of the sea nor of the large difference in gradient along the east and west coast. It does not give either any ides of the variation in width of the currents across the channels and of the transverse currents.

All this is the task of the two dimensional theory.

TWO DIMENSIONAL THEORY

Summary

We will study the two dimensional motion in a sea made up of three rectangular basins of constant depth, one basin corresponding to the Labrador Sea, Davis Strait and Baffin Bay respectively. Such a drastic simplification of the features of the domain under study is necessary in order to avoid excessive analytic complication and even in this very simple model one is faced with elaborate calculations.

We shall first derive the solution of the set of three partial differential equations which describe u, v, and Z in a rotating rectangular sea. Then we shall simplify this solution with the help of some approximations which hold well in the three seas under study; a sea satisfying these simplifying assumptions will be called narrow, deep and elongated (from now on denoted as a nde sea).

Before attempting to state and satisfy the boundary conditions that the solution has to satisfy in the model, we will introduce the concept of linear space. With the help of this concept we shall express the solution in terms of basic vectors which describe it fully in various spaces.

Once the solution is expressed in terms of the basic vectors it will be possible to understand more clearly its behaviour over the body of the sea and in the immediate vicinity of the boundaries.

We shall elucidate some statements by Taylor about the reflection of Kelvin waves in a rotating rectangular sea and we shall show that in a sea of finite dimensions and which is not too shallow, perfect reflection is always possible.

We will see as well that in a nde sea, only the part of the current which is constant across the mouth will be transmitted to the next boundary; the part of the current which is a function of x will only affect the first sea and will not reach the next one.

Once this is established we shall be in a position to state and satisfy the boundary conditions that the solution has to satisfy over our three basin model. In vector language this corresponds to equating two vectors which in turn implies that each of their components should be equal. This leads to a set of linear equations relating the undetermined constants contained in the solution thus allowing in principle a complete resolution of the problem.

With the help of the approximation of narrowness, depth and elongation we shall succeed in separating these linear equations into quasi independent subsets and eliminate most unknowns. Eventually we shall be faced with the task of inverting two 20 x 20 matrices.

Charts showing the elevation and currents will be drawn from the result of such calculations.

The Model

We use two different models for M_2 and K_1 ; the reasons for such a procedure are given below. The models are represented by the following quantities:

M_2

$l_1 = 1000$ Km	$b_1 = 762$ Km	$h_1 = 1570$ M
$l_2 = 200$ Km	$b_2 = 354$ Km	$h_2 = 370$ M
$l_2 = 1450$ Km	$b_3 = 439$ Km	$h_3 = 800$ M

K_1

$l_1 = 1000$ Km	$b_1 = 762$ Km	$h_1 = 1570$ M
$l_2 = 200$ Km	$b_2 = 177$ Km	$h_2 = 370$ M
$l_3 = 1150$ Km	$b_3 = 439$ Km	$h_3 = 800$ M

All the b and h quoted are averages of the observed b and h with the exception of b_2 for K_1.

In order to test the quality of the model we have subjected it to one dimensional computations for three channels of constant depth and width. Since it was important to preserve the position of the nodes where observation puts them, we had to make sea 3 (Baffin Bay) longer by 300 Km for M_2 in order to bring the node to 1660 Km. Within this model a transport of 2.6×10^5 m^3/sec is computed to take place at 2350 Km which compares favourably with the value of 2.5×10^5 m^3/sec assumed in the one dimensional theory (For $Z_o = 1$ Cm).

For K_1 the position of the node depends very sensitively on the variation of width in Davis Strait and the only way to bring the node to 1040 km with the above model is to cut b_2 by one half.

In both models perfect reflection is assumed to take place at the head.

An illustration of the model is given on the next page.

The Solution of the Equations of Hydrodynamics for a Rotating Rectangular Sea of Constant Depth

The equations of hydrodynamics in two dimensions in a Cartesian frame of reference rotating at an angular speed ω are (Lamb [7]).

$$\frac{\partial u}{\partial t} - 2\omega v = -g\frac{\partial Z}{\partial x} \qquad (7)$$

$$\frac{\partial v}{\partial t} + 2\omega u = -g\frac{\partial Z}{\partial y} \qquad (8)$$

$$\frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = -\frac{\partial Z}{\partial t} \qquad (9)$$

(7) and (8) express Newton's second law in the x and y direction and (9) states the conservation of mass. No external forces are acting on the fluid and the density is taken to be equal to 1.

The time variation of the variables u, v and Z is written as $e^{i\sigma t}$ where i is the imaginary unit. The real part of the variable is its value at time 0; its imaginary part, its value 1/4 period after. All the numbers we will handle from now on will be complex.

If in the basin under study, the depth is constant, (7), (8) and (9) take the form

$$i\sigma u - 2\omega v = -g\frac{\partial Z}{\partial x} \qquad (10)$$

$$i\sigma v + 2\omega u = -g\frac{\partial Z}{\partial y} \qquad (11)$$

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\sigma Z = 0 \qquad (12)$$

This is a set of simultaneous partial differential equations for the unknowns u, v and Z, and the basin over which they prevail is described by the equations

$$x = \pm a \qquad y = \pm L$$

The basin is therefore rectangular, of width 2a and length 2L with the origin being taken at its center. The basin is always closed on its sides i. e. at $x = \pm a$, but it may be open or closed at its extremities.

We need out of the general solution of (10) to (12) the special solution satisfying the boundary condition

$$u(\pm a) = 0 \qquad (14)$$

The set (10) to (12) admits of a solution satisfying

$$u' = 0 \qquad (15)$$

everywhere, therefore satisfying (14) a fortiori. (15) leads to three relations joining v and Z; these relations are consistent and therefore yield the solution

$$u' = 0 \qquad (16)$$

$$v' = A_o' \sinh[\lambda' x - iK_o' y] + B_o' \cosh[\lambda' x - iK_o' y] \qquad (17)$$

$$Z' = \left(\frac{h}{g}\right)^{\frac{1}{2}} \{ A_o \cosh[\lambda' x - iK_o' y] + B_o \sinh[\lambda' x - iK_o' y] \} \qquad (18)$$

$$\text{where } \lambda' \equiv \frac{2\omega}{(gh)^{\frac{1}{2}}} \text{ and } K_o' \equiv \frac{\sigma}{(gh)^{\frac{1}{2}}}$$

Ao, Bo are arbitrary constants.

This is the Kelvin solution and it would have been lost had we tried to obtain independent partial differential equations for each variable. Such a solution is not the only one satisfying equations (10) to (12) and the boundary condition (14); there exist a full infinity of extra solutions. But we shall soon see that the Kelvin solution describes the preponderant part of the motion in a sea of the dimensions described in the previous paragraphs.

To obtain the remaining solutions we eliminate v and Z; the equation satisfied by u is

$$(\nabla^2 + k^2)\, u = 0 \qquad (19)$$

(23) and (24) have solutions of the form

$$X \sim e^{\pm ij'x}$$

$$Y \sim e^{\pm K_j \left(\frac{\pi y}{2a}\right)}$$

where

$$K_j \equiv \left\{ j^2 - k^2 \cdot \left(\frac{2a}{\pi}\right)^2 \right\}^{\frac{1}{2}}.$$

The notation is somewhat simplified if we redefine the y variable as

$$\frac{\pi y}{2a} \to y$$

The interval over which it is defined is $-\frac{\pi L}{2a} \leqslant y \leqslant \frac{\pi L}{2a}$

and we set

$$\frac{\pi L}{2a} \equiv 1 \qquad l$$

a dimensionless number.

Then the y solution takes the form

$$Y \sim e^{\pm K_j y}$$

It is not convenient to renormalize the x coordinate in the same way; this will eventually become obvious.

The solution for u has the form

$$u \sim e^{\pm K_j y} e^{\pm ij'x}$$

By suitable linear combinations of such solutions we get expressions of the form

$$u \sim e^{\pm K_j y} \begin{Bmatrix} \sin j'x \\ \cos j'x \end{Bmatrix}$$

u will satisfy (14) if

j = an even integer for the sine solution

j = an odd integer for the cosine solution

The general solution of (19) is then

$$u = \sum_{\substack{m=1\\ odd}}^{\infty} \left(C'_m e^{K_m(y-l)} + D'_m e^{-K_m(y+l)} \right) \cos m' x \qquad (25)$$

$$+ \sum_{\substack{n=2\\ even}}^{\infty} \left(A'_n e^{K_n(y-l)} + B'_n e^{-K_n(y+l)} \right) \sin n' x$$

The primed arbitrary constants are related to the unprimed constants by the relations

$$C'_m \equiv \left\{ \frac{4\omega^2 + ghm'^2}{ghm K_m} \right\} \cdot \left(\frac{2a}{\pi} \right)^2 \cdot C_m \qquad \text{etc.}$$

From now on m will always denote an odd integer, n an even integer and j any integer.

A_n, B_n, C_m, D_m, form a two fold infinity of arbitrary constants and can be written out as

$$C_1,\ A_2,\ C_3,\ A_4,\ \ldots.$$

$$D_1,\ B_2,\ D_3,\ B_4,\ \ldots.$$

We have introduced the exponent in $K_j l$ in order to eliminate any exponential factors from the linear relations linking the arbitrary constants that will arise when further boundary conditions will be imposed. The solutions corresponding to an odd integer are of an even character and those corresponding to an even integer are of an odd character. In future we will often refer to a solution affected by an arbitrary constant as the wave pertaining to that constant, thus:

"The A_o Kelvin wave, the C_m wave etc."

Out of (25) we can derive the corresponding solutions for v and Z using relations (10) to (12). These are, including the Kelvin solution:

$$v = A_o \sinh[\lambda' x - iK_o(y-1)] + B_o \cosh[\lambda' x - iK_o(y-1)]$$

$$+ \sum_{m=1}^{\infty} \left(C_m e^{K_m(y-1)} - D_m e^{-K_m(y+1)} \right) \sin m' x$$

$$- \sum_{n=2}^{\infty} \left(A_n e^{K_n(y-1)} - B_n e^{-K_n(y+1)} \right) \cos n' x$$

$$+ i\,\Omega \left\{ \sum_{m=1}^{\infty} \left(C_m e^{K_m(y-1)} + D_m e^{-K_m(y+1)} \right) \frac{\cos m' x}{mK_m} + \sum_{n=2}^{\infty} \left(A_n e^{K_n(y-1)} + B_n e^{-K_n(y+1)} \right) \frac{\sin n' x}{nK_n} \right\} \quad (26)$$

$$Z = \left(\frac{h}{g}\right)^{\frac{1}{2}} \{A_o \cosh[\lambda' x - iK_o(y-1)] + B_o \sinh[\lambda' x - iK_o(y-1)]\}$$

$$-2\omega\alpha \left\{ \sum_{m=1}^{\infty} \frac{\cos m' x}{m} \left(C_m e^{K_m(y-1)} - D_m e^{-K_m(y+1)} \right) + \sum_{n=2}^{\infty} \frac{\sin n' x}{n} \left(A_n e^{K_n(y-1)} - B_n e^{-K_n(y+1)} \right) \right\}$$

$$- i\sigma\alpha \left\{ \sum_{m=1}^{\infty} \frac{\sin m' x}{K_m} \left(C_m e^{K_m(y-1)} + D_m e^{-K_m(y+1)} \right) - \sum_{n=2}^{\infty} \frac{\cos n' x}{K_n} \left(A_n e^{K_n(y-1)} + B_n e^{-K_n(y+1)} \right) \right\} \quad (27)$$

$$\Omega \equiv \frac{2\omega\sigma}{gh} \cdot \frac{4a^2}{\pi^2} \qquad \alpha \equiv \frac{2a}{\pi g} \qquad K_o \equiv \frac{2a}{\pi} K_o'$$

(25) to (27) represent a rigorous and complete solution of the set of partial differential equations (10) to (12) satisfying the boundary condition (14). The two-fold infinity of arbitrary constants they contain will be determined by applying extra boundary conditions at $y = \pm 1$.

The solution of Helmholtz equation satisfying boundary condition (14) exists only if the arbitrary constant j we introduced is an integer. Such values of j are eigenvalues and the solutions for u corresponding to j = 1, 2, 3, ..., are eigenfunctions. The solutions for u, v and Z corresponding to these integral values are usually called the Poincaré waves.

The Kelvin solution can be considered as the u eigenfunction corresponding to j = 0. It is the equivalent of the solution we obtained in Appendix 1 for a channel of constant width and depth; this can be seen by letting $\lambda' \to 0$. The $e^{\pm\lambda' x}$ factor brings in the two dimensional features which are the sloping of the surface across the direction of motion and the variation in v necessary to balance this elevation.

The Poincaré waves exist only in two dimensions and would vanish completely in one dimensional motion.

Two Dimensional Motion in a Narrow Deep Elongated Sea of Constant Depth

Although (25) to (27) represent the motion in an exceedingly simple basin they are nevertheless quite complicated and not easy to manipulate. We will now use the features of the three basins under study, their narrowness, depth and length to simplify the v and Z solutions.

Algebraically a nde sea is characterized by the property

$$\frac{\sigma^2}{gh} \cdot \left(\frac{2a}{\pi}\right)^2 \ll 1 \qquad (28)$$

which is equivalent to the statement that half the wave length Λ is large compared to the width

$$\left[\frac{2a}{\Lambda/2}\right]^2 \ll 1$$

For seas 1, 2 and 3, (28) has the value

Sea	M_2	K_1
1	.076	.020
2	.070	.005
3	.049	.013

Criterion (28) is therefore satisfied for the three seas.

(28) implies

1) $K_j \sim j$ for the sea under study.

By definition

$$K_j = \left\{ j^2 - \frac{\sigma^2 - 4\omega^2}{gh} \cdot \left(\frac{2a}{\pi}\right)^2 \right\}^{\frac{1}{2}}$$

The domain under study extends from latitude 55°N to 75°N. for M_2 $\sigma \sim 2\omega$ while for K_1, $\sigma \simeq \frac{1}{3}$ to $\frac{1}{4}$ of 2ω.

So that for all three basins

$$\frac{\sigma^2-4\omega^2}{gh}\left(\frac{2a}{\pi}\right)^2 \sim 0 \qquad \text{for } M_2$$

$$\frac{\sigma^2-4\omega^2}{gh}\left(\frac{2a}{\pi}\right)^2 \sim -.01 \qquad \text{or less for } K_1$$

But we will still denote K_j by its own symbol in the work that will follow in order to keep track of this quantity.

2) $$\Omega \ll 1$$

Referring to the definition of Ω the above statement is obvious since $2\omega \sim \sigma$ for both constituents. Explicit calculations give

Ω		
Sea	M_2	K_1
1	.07	.03
2	.06	.01
3	.04	.02

We shall as a consequence neglect Ω over the whole body of the sea. This means that we may drop the latter part of the solution for v; this is a very important simplification since it removes the coupling that would appear between the (A_n, B_n) and (C_m, D_m) constants when we try to satisfy the v boundary conditions. This approximation will lead to distinct sets of linear equations for the unknowns A_n, C_m, B_n, D_m.

No such simplification is possible for Z; all the Poincaré waves are of the same order of magnitude and the expression for Z has to be carried with all its complexity.

It will be convenient in two dimensions to impose boundary conditions on v and avoid manipulating Z as much as it is possible.

Now we wish to use the elongation of the sea in order to simplify our work further.

The Poincaré waves are all affected by exponents of the form

$$e^{\pm K_j(y\mp l)} \simeq e^{\pm j(y\mp l)}$$

so that at $y = \pm l$ we will have terms of the form

$$a_j e^{\mp 2jl} \pm b_j \quad , \quad a_j \pm b_j e^{\mp 2jl}$$

At both boundaries one of the arbitrary constants is affected by an exponent

$$e^{\mp 2jl}$$

If jl is large enough (the sea is elongated) the constant affected by this exponent may be neglected and in this fashion the Poincaré waves are separated into two distinct groups: (B_n, D_m) prevailing at the lower boundary and (A_n, C_m) at the upper boundary. The A_0 and B_0 waves are not affected by this factor and will be present at both boundaries; they will transmit the motion from one set of Poincaré waves to the other.

Let us now verify that the three basins are elongated:

e^{-2jl}				
Sea	j = 1		j = 2	
	M_2	K_1	M_2	K_1
1	.016	.016	.000	.000
2	.170	.029	.029	.001
3	.000	.003	.000	.000

This assumption is not justified for M_2, $j = 1$ in Sea 2. We will take care of this in our calculations.

The above shows that a Poincaré wave for which the exponential factor has value 1 at the boundary will be affected by a factor which will damp it rapidly away from the boundary. The Kelvin waves are unaffected by such an exponent and their amplitude is modulated trigonometrically throughout the length of the sea. Therefore irrespective of the relative magnitude of the Kelvin and Poincaré waves at the boundary, the Kelvin wave only will eventually survive as one moves up the sea. As a consequence, beyond a certain point, the cross current u becomes negligible and v and Z are determined almost exclusively by the amplitude of the Kelvin wave. In order to re-create more Poincaré waves one needs to introduce further boundaries into the path of the Kelvin wave: a wall, a constriction or a change in depth. At such a point the presence of Poincaré waves is essential to balance the Kelvin waves. The effect of these Poincaré waves will be felt ahead of the boundary but again not very far away from it.

In the case of co-oscillation, when the energy is introduced at the mouth of the sea, this energy will be distributed to the Kelvin and Poincaré waves; indeed it is necessary as a rule to have both kinds of waves in order to represent the motion at the mouth. However only the Kelvin wave will move unimpeded up the sea and therefore the portion of the energy which has been assigned to it will travel with it.

We summarize these remarks in two theorems:

Theorem 1. In a nde sea, the Poincaré waves are significant only in the vicinity of the boundaries.

Corollary. The current tends to become unidirectional away from the boundaries.

Theorem 1 and its Corollary may be restated in more poetic language as: In a nde the Poincaré waves tune themselves in only in the vicinity of the boundaries while the Kelvin waves control the motion over the body of the sea.

Theorem 2. In a nde sea, in the case of co-oscillation, only the contribution to the Kelvin wave will propagate far into the sea.

Corollary. Only the part of the energy which is transmitted to the Kelvin wave will propagate into the sea.

We shall restate Theorem 2 and its corollary in a much stronger form for **v**, once we have introduced the concept of vector space.

The Concept of Functions as Vectors

Before we state the boundary conditions to be satisfied by v and Z at $y = \pm 1$ it is useful to bring in the concept of a linear vector space (Halmos [5]).

Geometrically a linear vector space consists of all the vectors which can be constructed from the basic vectors. The complete set of these basic vectors is called the base and any vector is made up of a sum of components directed along these basic vectors and of a length equal to a multiple of that of the basic vector.

Algebraically a linear space consists of the totality of linear combinations that can be formed out of a set of linearly independent quantities. The complete set of these linearly independent quantities is the base and their number gives the dimension of the space.

If $(\underline{b}_1, \underline{b}_2, \ldots \underline{b}_N)$ form the base of a vector space, any vector $\underline{V}$ in that space is a linear combination of these basic vectors:

$$\underline{V} = \sum_{j=1}^{N} a_j \underline{b}_j$$

where the a_j's are real numbers.

The $\underline{b}_j$'s satisfy the orthogonal property

$$\underline{b}_j \cdot \underline{b}_k = 1_j \delta_{jk}$$

where 1_j is the length of the j^{th} basic vector and δ_{jk} is the Kronecker delta symbol. The dot product is as yet a purely symbolic operation.

Orthogonality insures linear independence since

$$\sum_{j=1}^{N} c_j \underline{b}_j = 0$$

implies $c_j = 0$ for all j's.

We deduce

$$\underline{V} \cdot \underline{b}_j = a_j.$$

from the above relations without any need to go further into the significance of the dot product.

An example of a linear space is the three dimensional Euclidean space which is spanned by the three basic vectors $(\underline{b}_1, \underline{b}_2, \underline{b}_3)$ satisfying

$$\underline{b}_j \cdot \underline{b}_k = \delta_{jk}$$

If we denote the $\underline{b}_j$'s by the triads

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1),$$

then any vector $\underline{V} = (a_1, a_2, a_3)$ can be written as

$$\underline{V} = \sum_{j=1}^{3} a_j \underline{b}_j$$

This notation suggests the definition of the dot product by

$$\underline{V} \cdot \underline{U} \equiv \sum_{l=1}^{3} V_l U_l$$

since this would insure

$$\underline{b}_1 \cdot \underline{b}_2 = 1 \times 0 + 0 \times 1 + 0 \times 0 = 0$$
$$\underline{b}_1 \cdot \underline{b}_1 = 1 \times 1 + 0 \times 0 + 0 \times 0 = 1 \qquad \text{etc.}$$

and

$$\underline{V} \cdot \underline{b}_j = a_1 \delta_{1j} + a_2 \delta_{2j} + a_3 \delta_{3j} = \begin{cases} a_1 & \text{if } j = 1 \\ a_2 & \text{if } j = 2 \\ a_3 & \text{if } j = 3 \end{cases}$$

Geometrically the dot product represents the projection of one vector along the other; algebraically it can be considered as a summation over a variable. Thus

$$\underline{b}_j \cdot \underline{b}_k = \sum_{l=1}^{3} b_{jl} b_{kl}$$

For N dimensions, 3 would be replaced by N but 1 could still be interpreted as a direction in the geometric sense.

A natural generalization of the dot product is

$$\underline{b}_j \cdot \underline{b}_k = \int_\alpha^\beta b_j(x)\, b_k(x)\, dx$$

where x is no longer interpreted as a direction but simply as a continuous variable defined over the interval (α, β). Such a definition of the dot product satisfies all the formal properties that one needs of a dot product, two vectors being orthogonal when

$$\int_\alpha^\beta b_j(x)\, b_k(x)\, dx = l_j\, \delta jk$$

With such a definition we conceive of a linear vector space as the totality of the functions that can exist over the interval (α, β). Such a space has a base $(s_1, s_2, s_3, \ldots)$ where N can be taken to be ∞ if this is necessary, so that any function can be written as

$$f(x) = \sum_{j=1}^{\infty} a_j S_j(x)$$

or, in vector notation

$$\underline{f} = \sum_{j=1}^{\infty} a_j \underline{S}_j$$

Functions and vectors having been shown to be formally identical we shall conceive of a function as a vector each time this happens to be useful. We state

1) Any vector (function) $\underline{f}$ in a given space can be expressed in terms of a set of basic vectors s_j's:

$$\underline{f} = \sum_{j=1}^{N} a_j \underline{S}_j$$

2) The component a_j is given by

$$\alpha_j = \int_{-a}^{a} f(x)\ S_j(x)\ dx$$

if the space is defined over the interval

$$-a \leqslant x \leqslant a$$

3) Two vector $\underline{f}$ and $\underline{g}$ are equal only if each of their components are equal:

We know we may write

$$\underline{f} = \Sigma\ \alpha_j \underline{S}_j \qquad \underline{g} = \Sigma\ c_j \underline{S}_j$$

If

$$\underline{f} = \underline{g}$$

this involves

$$\alpha_j = c_j \qquad j = 1,\ 2,\ 3,\ \ldots.$$

We now wish to apply the above notions to our problem. Solving for u we had obtained the solution in term of the functions $(\sin n'x,\ \cos m'x)$.

This set of functions forms the base of a space consisting of all the functions, continuous of discontinuous, defined over $-a \leqslant x \leqslant a$ and which vanish at $\pm$ a. This space contains in particular the constant 0 but none other.

The basic vectors

$$(\sin n'x,\ \cos m'x)$$

are orthogonal

$$\int_{-a}^{a} \sin n'x \sin N'x\ dx = a\ \delta_{nN}$$

$$\int_{-a}^{a} \cos m'x \cos M'x\ dx = a\,\delta_{mM}$$

$$\int_{-a}^{a} \sin n'x \cos m'x\ dx = 0$$

and are therefore linearly independent.

But this space is too small for our purposes; it contains u since u satisfies

$$u(\pm a) = 0$$

but it cannot contain v and Z which do not necessarily vanish at $\pm$ a.

We need a space that contains all functions, continuous or discontinuous, defined over $-a \leqslant x \leqslant a$. This space containing all functions defined over the same interval and which vanish at $\pm$ a is a subspace of that space and is completely embodied into it.

The set $(\cos n'x, \sin m'x)$ forms a base of this larger space. These vectors are orthogonal

$$\int_{-a}^{a} \cos n'x \cos N'x \, dx = a_n \, \delta_{nN} \qquad a_n \equiv \begin{cases} \frac{a}{2} & n = o \\ a & n \neq o \end{cases}$$

$$\int_{-a}^{a} \sin m'x \sin M'x \, dx = a \delta_{mM}$$

$$\int_{-a}^{a} \sin m'x \cos n'x \, dx = 0$$

and every function can be expressed in terms of these basic vectors. In particular this space contains all the constants, including 0. It contains as well the set $(\cos m'x, \sin n'x)$ These are given in terms of the basic vectors of the larger space by

$$\cos m'x = \sum_{n=0}^{\infty} P_{mn} \cos n'x$$

$$\sin n'x = \sum_{m=1}^{\infty} Q_{nm} \sin m'x$$

$$P_{mn} \equiv \frac{4}{\pi} (-)^{\frac{m-1}{2}} \cdot \begin{cases} \dfrac{1}{2m} & n = 0 \\ \dfrac{(-)^{n/2} m}{m^2 - n^2} & n \neq 0 \end{cases}$$

$$Q_{nm} \equiv \frac{4}{\pi} (-)^{\frac{n+m-1}{2}} \frac{n}{m^2 - n^2}$$

The base we shall use is then

$$(\cos n'x, \quad \sin m'x) \qquad n = 0, 2, 4, \quad \ldots$$
$$m = 1, 3, 5, \quad \ldots$$

Finally we must remember that we have three intervals over which we wish to define a space, one for each basin. And when we come to satisfy the boundary conditions at $y = \pm\ l_i$, $i = 1, 2, 3$, all the functions will be at the boundary between these various intervals. We must therefore be able to express any function in terms of the basic set that happens to be convenient; but we may only move from a larger space to its subspace. In our case space 1 is the largest, space 2 the smallest. The base of 2 is contained in space 1 or 3.

This restricts the expansion of a function defined over interval 1 or 3 in terms of base 2; we can do this only if we consider its part which is defined over 2. This will be the case for Z where we wish to insure that

$Z_1\ (+\ l_1)$ and $Z_3\ (-\ l_3)$ are equal to $Z_2\ (-\ l_2)$ and $Z_2(+\ l_2)$ respectively

over $-a_2 \leqslant x \leqslant a_2$, the variation of Z_1 and Z_3 over the remaining part of the interval being left undetermined.

Discontinuous functions (having finite discontinuities) are contained in any of the three spaces; $v_1\ (+\ l_1)$ and $v_3\ (-\ l_3)$ are discontinuous but still they are fully contained in their respective spaces.

The Solution for v in Terms of the Basic Vectors: Some Important Remarks

Although we have derived the solution for v and Z from that of u, we will handle only v and Z from now on.

It is neither convenient nor necessary to expand v and Z in terms of their basic vectors in 1, 2, or 3. But in order to gain further insight into our solution we shall write the solution for v without reference to any sea in terms of the basic vectors. This is

$$V = \left[B_o \cos K_o (y-1) - i A_o \sin K_o (y-1)\right] \sum_{n=0}^{\infty} p_n \cos n'x$$

$$+ \left[-i B_o \sin K_o (y-1) + A_o \cos K_o (y-1)\right] \sum_{m=1}^{\infty} q_m \sin m'x$$

$$+ \sum_{m=1}^{\infty} \left(C_m e^{K_m(y-1)} - D_m e^{-K_m(y+1)} \right) \sin m'x + i\,\Omega \sum_{m=1}^{\infty} \sin m'x \sum_{n=2}^{\infty} \frac{Q_{nm}}{nK_n} \left(A_n e^{K_n(y-1)} + B_n e^{-K_n(y+1)} \right)$$

$$- \sum_{n=2}^{\infty} \left(A_n e^{K_n(y-1)} - B_n e^{-K_n(y+1)} \right) \cos n'x + i\,\Omega \sum_{n=0}^{\infty} \cos n'x \sum_{m=1}^{\infty} \frac{P_{mn}}{mK_m} \left(C_m e^{K_m(y-1)} + D_m e^{-K_m(y+1)} \right) \qquad (29)$$

where (p_n, q_m) are the components of $(\cosh \lambda' x, \sinh \lambda' x)$; these are given on page 68 ff.

If we neglect the terms affected by the factor $i\,\Omega$ (nde sea) we note two things:

1) The Poinçaré waves are already in terms of the basic vectors,

2) The Kelvin waves only will contribute to the constant term.

The amplitude of the vector directed along the basic vector corresponding to $n = 0$ gives the part of the current which is constant across $-a \leqslant x \leqslant a$ for a given value of y. The higher modes corresponding to $j \geqslant 1$ represent the part of the current which varies with x across the section; in the actual expansion it is given by a superposition of even and odd modes which can represent any reasonable function of x.

If we choose a section across which (29) holds, to be the mouth of a sea in which there is motion due to a neighbouring sea, the current due to the external body of water can always be expanded in terms of the basic vectors $(\cos n'x, \sin m'x)$ and this allows the evaluation of one half of the double infinity of arbitrary

constants contained in (29) by equating the two vector expansions representing v at this section (i. e. we satisfy the boundary condition at the mouth). Theorem 1 states that the only constants which wont be affected by a very small factor at the mouth will be (A_o, B_o, A_n, C_m). Identifying the components of the vectors and choosing for the moment the mouth to lie at $y = +1$ (in our actual model it will lie at $y = -1$), we determine B_o explicitely in terms of the constant value of the current while A_n and C_m will be given in terms of the higher modes and B_o or A_o.

When we come to the next boundary to satisfy the extra boundary condition that will allow the evaluation of the latter half of the double infinity of arbitrary constants, A_n and C_m will have been damped out and since they contain the higher harmonics of v this means that these higher harmonics do not affect at all the solution at the next boundary and that (A_o, B_n, D_m) will be given exclusively in terms of the constant part of the current. Once this boundary condition is satisfied (assuming the current or the elevation to be given explicitely there), (29) is given explicitely in terms of the boundary values.

At the mouth the motion will embody all the higher modes of v as well as the fundamental one but at the next boundary the v solution will depend exclusively on the constant value of the current across the mouth while the higher harmonics of this same current are not involved at all. So that at some distance beyond the mouth the solution is completely indifferent to the part of the current which is function of x across the mouth and it depends mainly on the part which is constant across the initial section.

We may therefore restate Theorem 2 and its Corollary in a much stronger form for v:

<u>Theorem: The part of the current which is constant across the mouth will propagate up a nde sea; what is function of x will be damped.</u>

This theorem is intuitive when we realize that a nde sea in the intermediate between a channel of constant depth and a flat rectangular sea. We shall pursue further this analogy by proving later in the thesis that satisfying the boundary conditions for $n = 0$ for the Kelvin waves only and neglecting all the Poincaré waves in a nde sea is equivalent to a zero order of approximation to satisfying the boundary conditions in a channel of constant depth.

One of our basic assumption is that there is perfect reflection of the head; we wish to see if this is always possible.

By perfect reflection we mean that at the mouth and beyond the mouth $y \leqslant -1$, (we return to our previous choice for the position of the mouth) the motion should be given by

$$V = \mathcal{a}_o \sinh[\lambda' x - iK_o(y-l)] \qquad (30)$$

(30) represents the superposition of two Kelvin waves of equal amplitude $\mathcal{a}$ and travelling in opposite directions. We admit of a "zone of confusion" within the sea but not beyond $y < -l$.

Using the solution for v in a nde sea and applying the boundary conditions

$$v(+l) = 0$$

$$v(-l) = \mathcal{a}_o \sinh[\lambda' x + iK_o 2l]$$

leads to

$$B_o = 0 \qquad n = 0$$

$$B_o p_n - A_n = 0 \qquad n = 2,4,6\ldots$$

$$A_o q_m + C_m = 0 \qquad m = 1,3,5\ldots$$

at $y = +l$ which implies

$$B_o = 0 \qquad A_n = 0 \qquad C_m = -A_o q_m$$

and

$$iA_o \sin 2K_o l = i\mathcal{a}_o \sin 2K_o l \qquad n = 0$$

$$p_n(iA_o \sin 2K_o l) + B_n = p_n(i\mathcal{a}_o \sin 2K_o l) \qquad n = 2,4,6\ldots$$

$$A_o \cos 2K_o l \cdot q_m - D_m = \mathcal{a}_o \cos 2K_o l \qquad m = 1,3,5\ldots$$

at $y = -l$ so that

$$A_o = \mathcal{a}_o \qquad B_n = 0 \qquad D_m = 0$$

We conclude that in a nde sea

1) Perfect reflection always takes place,

2) There is no "zone of confusion": the motion is given everywhere within the sea and beyond by $\mathcal{a}_o \sinh[\lambda'_x - iK_o(y-l)]$ except in the immediate vicinity of the reflecting wall.

This conclusion is again intuitive since a nde sea is very closely related to a channel of constant depth where perfect reflection is always possible. But this seems to clash with Taylor's (15) conclusion that in a semi infinite channel Kelvin waves can be perfectly reflected only if the reflecting wall is located at some specific distance from a given origin.

The sea studied by Taylor does not satisfy the restrictive assumption of narrowness and depth; but by definition it is strictly elongated since the mouth is assumed to lie at $y = -\infty$

Let us for the moment take the mouth at $y = \quad 1'$, $1'$ being some large number.

The condition $v(+1) = 0$, using the full v solution gives

$$B_o p_o + i\Omega \sum_{m=1}^{\infty} \frac{P_{mo}}{mK_m} C_m = 0 \qquad n = 0$$

$$B_o p_n - A_n + i\Omega \sum_{m=1}^{\infty} \frac{P_{mn}}{mK_m} C_m = 0 \qquad n = 2,4,6\ldots$$

$$A_o q_m + C_m + i\Omega \sum_{n=2}^{\infty} \frac{Q_{nm}}{nK_n} A_n = 0 \qquad m = 1,3,5\ldots \qquad (31)$$

We are no longer allowed to set $B_o = 0$ nor $A_o = a_o$ beforehand. These two unknowns should be evaluated by using the boundary condition at $-1'$ because then we would have $2x\ \infty$ equations in $2x\ \infty$ unknowns.

If we insist on setting $B_o = 0$, $A_o = a_o$, the lower boundary condition will yield

$$B_n = D_m = 0$$

i.e. the motion is purely given by a_o inside and outside the sea. This is the situation wished and described by Taylor. At the same time it leaves the system of equations (31) overdeterminate since we have $1x\ \infty$ relations in $1x\ \infty\ -1$ unknowns. Unless the determinant of this system of equations vanishes there is no other solution but the identically zero solution:

$$A_o = C_m = A_n = 0$$

which contradicts the assumption

$$A_o = a_o$$

There is no reason at all for the determinant of (31) to vanish unless we introduce into it some parameter. The obvious choice is 1.

We transform the coordinates so that the origin lies at -1; the head lies at $y = +21$

The system (31), assuming $B_o = 0$, $A_o = a_o$, becomes

$$p_o a_o \sin K_o l - \Omega \sum_{m=1}^{\infty} \frac{P_{mo}}{mK_m} C_m = 0$$

$$p_n a_o \sin K_o l - A_n - \Omega \sum_{m=1}^{\infty} \frac{P_{mn}}{mK_m} C_m = 0$$

$$q_m a_o \cos K_o l + C_m + \Omega \sum_{n=2}^{\infty} \frac{Q_{nm}}{nK_n} A_n = 0$$

We redefined the arbitrary constant A_n to be iA_n $(n \neq o)$.

The determinant of the coefficients is

$$\begin{array}{c|ccc|}
 & 0 & \text{odd} & \text{even} \\
0 & p_o a_o \sin K_o l & -\Omega \dfrac{P_{mo}}{mK_m} & 0 \\
\text{odd} & q_m a_o \cos K_o l & \delta_{mM} & \Omega \dfrac{Q_{nm}}{nK_n} \\
\text{even} & p_n a_o \sin K_o l & -\Omega \dfrac{P_{mn}}{mK_m} & -\delta_{nN}
\end{array}$$

Where 0 denotes column or row 0, m, odd columns and rows and n, even columns and rows, excluding 0.

If we insert the explicit expression for P_{mn}, Q_{mn}, p_n and q_m, and if we perform the following operations:

1) Remove the common factor a_o from column 0

2) Take as Taylor, $\lambda' a = \frac{\lambda\pi}{2}$

3) Multiply row 0 by the constant 2,

4) Remove the common factor $\frac{4}{\pi} \lambda \cos K_o l$ from column 0,

5) Multiply all odd rows and columns by $(-)^{\frac{m-1}{2}}$, all even rows and columns

by $(-)^{\frac{n}{2}}$.

6) Remove the common factor $\frac{\Omega}{K_j}\frac{4}{\pi}$ from all even and odd columns,

7) Multiply all odd columns by $\sinh\frac{\lambda\pi}{2}$, all even columns by $\cosh\frac{\lambda\pi}{2}$

8) Divide all odd rows by $\cosh\frac{\lambda\pi}{2}$, all even rows by $\sinh\frac{\lambda\pi}{2}$,

We obtain

$$\begin{array}{c|ccc|}
 & 0 & \text{odd} & \text{even} \\
0 & \frac{\operatorname{tg} K_0 l}{\lambda^2} & -\frac{1}{m^2} & 0 \\
\text{odd} & \frac{1}{\lambda^2+m^2} & \frac{4}{\pi}\cdot\frac{K_m}{\Omega}\cdot \operatorname{tgh}\frac{\lambda\pi}{\lambda}\cdot\delta_{mM} & \frac{1}{m^2-n^2} \\
\text{even} & \frac{\operatorname{tg} K_0 l}{\lambda^2+n^2} & \frac{1}{n^2-m^2} & -\frac{4}{\pi}\cdot\frac{K_n}{\Omega}\cdot\operatorname{cotgh}\frac{\lambda\pi}{2}\cdot\delta_{nN}
\end{array}$$

which is the determinant derived by Taylor but expressed in our own notation making due allowance to the fact that our a_o corresponds to $a_{o/i}$ in Taylor's notation and that our frame of reference is left handed while his is right handed. Such a determinant gives the values of l for which the system (31) will have a solution that can be expressed in terms of a_o only.

As long as l ' is finite, l is a paradoxical quantity since it is found in a way that makes it completely independent of l ' and it depends exclusively on our choice of an origin.

l can be meaningful only if $l' \to \infty$; then we wish to insure that a_o prevails over the whole of the semi infinite channel.

If l' is finite, l is meaningless and we need to solve the full $2x\,\infty$ linear equations to extract values for A_o and B_o. In this case A_o is different from a_o but the Poincaré waves B_n and D_m are such that they balance A_o and B_o to produce a_o at the mouth.

Perfect reflection takes place as long as K_j is real for all j's. But over the sea, between the mouth and the head, there is a "zone of confusion" where B_o exists along A_o.

As the sea becomes more and more nde, $\Omega \to 0$ $B_o \to 0$ and $A_o \to a_o$.

The above considerations allow us to state the general theorem:

Theorem: If K_j is real for all j's, a Kelvin wave can always be perfectly reflected in a rotating rectangular sea of constant depth of any length.

The Boundary Conditions

At $y_1 = -l_1$

$$v_1(-l_1) = f(x) = \sum_{j=0}^{\infty} \theta_j S_j(x) \qquad -a_1 \leqslant x \leqslant a_1 \tag{32}$$

At $y_1 = +l_1$, $y_2 = -l_2$

$$v_1(+l_1) \cdot \frac{h_1}{h_2} = \begin{cases} 0 & a_2 < |x| \leqslant a_1 \\ v_2(-l_2) & -a_2 \leqslant x \leqslant a_2 \end{cases} \tag{33}$$

$$Z_1(+l_1) = Z_2(-l_2) \qquad -a_2 \leqslant x \leqslant a_2 \tag{34}$$

At $y_2 = +l_2$, $y_3 = -l_3$

$$v_3(-l_3) \cdot \frac{h_3}{h_2} = \begin{cases} 0 & a_2 < |x| < a_3 \\ v_2(+l_2) & -a_2 \leqslant x \leqslant a_2 \end{cases} \tag{35}$$

$$Z_3(-l_3) = Z_2(+l_2) \qquad -a_2 \leqslant x \leqslant a_2 \tag{36}$$

At $y_3 = +l_3$

$$v_3(+l_3) = 0 \qquad -a_3 \leqslant x \leqslant a_3 \tag{37}$$

The boundary condition at the mouth has already been discussed.

The v boundary condition at the junctions of the basins indicates that $v_1\ (+l_1)$ and $v_3\ (-l_3)$ are equal to a vector with the properties described on the right hand side. This will involve evaluating such a vector in space 1 and 3. The Z boundary condition restricts the value of Z_1 and Z_3 only over the interval $-a_2 \leqslant x \leqslant a_2$; we then have to express $Z_1\ (+l_1)$ and $Z_3\ (-l_3)$ in terms of the basic vectors in space 2 and equate them to $Z_2\ (\mp l_2)$.

We therefore need

1) $v_1\ (+l_1)$ and $v_3\ (+l_3)$ in their respective spaces.

2) A vector which we shall denote by $V\ (+l_2)$ with the properties described by (33) and (35) in space 1 and 3.

3) $Z_1\ (+l_1)$, $Z_3\ (-l_3)$ and $Z_2\ (\pm l_2)$ in space 2.

These quantities are, using solutions (26) and (27), for k = 1 or 3:

$$v_k(\pm l_k) = \begin{bmatrix} B_o^{(k)} \\ B_o^{(k)} \cos 2K_o^{(k)} l_k + i A_o^{(k)} \sin 2K_o^{(k)} l_k \end{bmatrix} \cdot \sum_{n_k=0}^{\infty} p_{n_k}^{(k)} \cos n_k' x$$

$$+ \begin{bmatrix} A_o^{(k)} \\ A_o^{(k)} \cos 2K_o^{(k)} l_k + i B_o^{(k)} \sin 2K_o^{(k)} l_k \end{bmatrix} \cdot \sum_{m_k=1}^{\infty} q_{m_k}^{(k)} \sin m_k' x$$

$$+ \sum_{n_k=1}^{\infty} \cos n_k' x \begin{pmatrix} -A_{n_k}^{(k)} \\ B_{n_k}^{(k)} \end{pmatrix} + \sum_{m_k=1}^{\infty} \sin m_k' x \begin{pmatrix} C_{m_k}^{(k)} \\ -D_{m_k}^{(k)} \end{pmatrix} \tag{38}$$

$$V(\pm l_2) = \begin{bmatrix} B_o^{(2)} \\ B_o^{(2)} \cos 2K_o^{(2)} l_2 + i A_o^{(2)} \sin 2K_o^{(2)} l_2 \end{bmatrix} \cdot \sum_{n_k=0}^{\infty} r_{n_k}^{(2)} \cos n_k' x$$

$$+ \begin{bmatrix} A_o^{(2)} \\ A_o^{(2)} \cos 2K_o^{(2)} l_2 + i B_o^{(2)} \sin 2K_o^{(2)} l_2 \end{bmatrix} \cdot \sum_{m_k=1}^{\infty} t_{m_k}^{(2)} \sin m_k' x$$

$$+ \frac{a_2}{a_k} \left\{ \sum_{n_k=2}^{\infty} \cos n_k' x \sum_{n_2=2}^{\infty} P_{n_k n_2} \begin{pmatrix} -A_{n_2}^{(2)} \\ B_{n_2}^{(2)} \end{pmatrix} + \sum_{m_k=1}^{\infty} \sin m_k' x \sum_{m_2=1}^{\infty} Q_{m_k m_2} \begin{pmatrix} C_{m_2}^{(2)} \\ -D_{m_2}^{(2)} \end{pmatrix} \right\} \tag{39}$$

$$
\begin{aligned}
Z_k(\pm l_k) = \left(\frac{h_k}{g}\right)^{\frac{1}{2}} \Bigg\{ & \begin{bmatrix} A_o^{(k)} \\ A_o^{(k)} \cos 2K_o^{(k)} l_k + i\, B_o^{(k)} \sin 2K_o^{(k)} l_k \end{bmatrix} \sum_{n_2=0}^{\infty} u_{n_2}^{(k)} \cos n_2' x \\
& + \begin{bmatrix} B_o^{(k)} \\ B_o^{(k)} \cos 2K_o^{(k)} l_k + i\, A_o^{(k)} \sin 2K_o^{(k)} l_k \end{bmatrix} \sum_{m_2=1}^{\infty} v_{m_2}^{(k)} \sin m_2' x \Bigg\} \\
- \alpha_k \Bigg\{ & \sum_{n_2=0}^{\infty} \cos n_2' x \left[-i\sigma \sum_{n_k=2}^{\infty} \frac{P_{n_k n_2}}{K_{n_k}} \begin{pmatrix} A_{n_k}^{(k)} \\ B_{n_k}^{(k)} \end{pmatrix} + 2\omega_k \sum_{m_k=1}^{\infty} \frac{P_{m_k n_2}}{m_k} \begin{pmatrix} C_{m_k}^{(k)} \\ -D_{m_k}^{(k)} \end{pmatrix} \right] \\
& + \sum_{m_2=1}^{\infty} \sin m_2' x \left[i\sigma \sum_{m_k=1}^{\infty} \frac{Q_{m_k m_2}}{K_{m_k}} \begin{pmatrix} C_{m_k}^{(k)} \\ D_{m_k}^{(k)} \end{pmatrix} + 2\omega_k \sum_{n_k=2}^{\infty} \frac{Q_{n_k m_2}}{n_k} \begin{pmatrix} A_{n_k}^{(k)} \\ -B_{n_k}^{(k)} \end{pmatrix} \right] \Bigg\}
\end{aligned}
\qquad (40)
$$

$$
\begin{aligned}
Z_2(\pm l_2) = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \Bigg\{ & \begin{bmatrix} A_o^{(2)} \\ A_o^{(2)} \cos 2K_o^{(2)} l_2 + i\, B_o^{(2)} \sin 2K_o^{(2)} l_2 \end{bmatrix} \sum_{n_2=0}^{\infty} p_{n_2}^{(2)} \cos n_2' x \\
& + \begin{bmatrix} B_o^{(2)} \\ B_o^{(2)} \cos 2K_o^{(2)} l_2 + i\, A_o^{(2)} \sin 2K_o^{(2)} l_2 \end{bmatrix} \sum_{m_2=1}^{\infty} q_{m_2}^{(2)} \sin m_2' x \\
- \alpha_2 \Bigg\{ & -i\sigma \sum_{n_2=2}^{\infty} \frac{\cos n_2' x}{K_{n_2}} \begin{pmatrix} A_{n_2}^{(2)} \\ B_{n_2}^{(2)} \end{pmatrix} + 2\omega_2 \sum_{n_2=0}^{\infty} \cos n_2' x \left[\sum_{m_2=1}^{\infty} \frac{P_{m_k n_2}}{m_2} \begin{pmatrix} C_{m_2}^{(2)} \\ -D_{m_2}^{(2)} \end{pmatrix} \right] \\
& + \sum_{m_2=1}^{\infty} \sin m_2' x \left[\frac{i\sigma}{K_{m_2}} \begin{pmatrix} C_{m_2}^{(2)} \\ D_{m_2}^{(2)} \end{pmatrix} + 2\omega_2 \sum_{n_2=2}^{\infty} \frac{Q_{n_2 m_2}}{n_2} \begin{pmatrix} A_{n_2}^{(2)} \\ -B_{n_2}^{(2)} \end{pmatrix} \right] \Bigg\}
\end{aligned}
\qquad (41)
$$

Where

$$p_{o}^{(k)} \equiv \frac{1}{2a_k}\int_{-a_k}^{a_k} \cosh\lambda'_k x \, dx = \frac{\sinh\,(\lambda'_k a_k)}{(\lambda'_k a_k)}$$

$$p_{n_k}^{(k)} \equiv \frac{1}{a_k}\int_{-a_k}^{a_k} \cosh\lambda'_k x \, \cos n'_k x \, dx = \frac{8}{\pi^2}\cdot(\lambda'_k a_k)\cdot(-)^{\frac{n_k}{2}}\cdot \frac{\sinh\,(\lambda'_k a_k)}{(\lambda_k^2 + n_k^2)}$$

$$q_{m_k}^{(k)} \equiv \frac{1}{a_k}\int_{-a_k}^{a_k} \sinh\,\lambda'_k x \, \sin m'_k x \, dx = \frac{8}{\pi^2}\cdot(\lambda'_k a_k)\cdot(-)^{\frac{m_k - 1}{2}}\cdot \frac{\cosh\,(\lambda'_k a_k)}{(\lambda_k^2 + m_k^2)}$$

$$r_{o_k}^{(2)} \equiv \frac{1}{2a_k}\int_{-a_2}^{a_2} \cosh\lambda'_2 x \, dx = \frac{\sinh\,(\lambda'_2 a_2)}{(\lambda'_2 a_k)}$$

$$r_{n_k}^{(2)} \equiv \frac{1}{a_k}\int_{-a_2}^{a_2} \cosh\lambda'_2 x \, \cos n'_k x \, dx = \frac{4}{\pi}\,\frac{\left\{\frac{\lambda_2 a_k}{a_2}\cdot \sinh\lambda'_2 a_2 \cdot \cos n'_k a_2 + n_k \cdot \cosh\lambda'_2 a_2 \cdot \sin m'_k a_2\right\}}{\left\{\left(\frac{\lambda_2 a_k}{a_2}\right)^2 + n_k^2\right\}}$$

$$t_{m_k}^{(2)} \equiv \frac{1}{a_k}\int_{-a_2}^{a_2} \sinh\lambda'_2 x \, \sin m'_k x \, dx = \frac{4}{\pi}\,\frac{\left\{\frac{\lambda_2 a_k}{a_2}\cdot \cosh\lambda'_2 a_2 \cdot \sin m'_k a_2 - m_k \sinh\lambda'_2 a_2 \cdot \cos m'_k a_2\right\}}{\left\{\left(\frac{\lambda_2 a_k}{a_2}\right)^2 + m_k^2\right\}}$$

$$P_{j_k 0 i} = \frac{1}{2a_i}\int_{-a_i}^{a_i} \cos j'_k x \, dx = \frac{2}{\pi}\cdot\frac{a_k}{a_i}\cdot\frac{\sin(j'_k a_i)}{j_k} = \frac{2}{\pi}\,\frac{(-)^{\frac{m-1}{2}}}{m} \quad \text{for} \begin{cases} i = k \\ j_k = m \end{cases}$$

$$P_{j_k n_i i} = \frac{1}{a_i}\int_{-a_i}^{a_i} \cos j'_k x \cos n'_i x \, dx = \frac{4}{\pi}\cdot\frac{a_k}{a_i}\cdot (-)^{\frac{n_i}{2}} \frac{j_k \sin(j'_k a_i)}{\left[j_k^2 - \left(\frac{n_i a_k}{a_i}\right)^2 \right]}$$

$$= \frac{4}{\pi}(-)^{\frac{n+m-1}{2}} \frac{m}{m^2 - n^2} \quad \text{for} \begin{cases} i = k \\ j_k = m \\ n_i = n \end{cases}$$

$$Q_{j_k m_i i} = \frac{1}{a_i}\int_{-a_i}^{a_i} \sin j'_k x \sin m'_i x \, dx = -\frac{4}{\pi}\cdot\frac{a_k}{a_i}\, (-)^{\frac{m_i-1}{2}} \cdot \frac{j_k \cos(j'_k a_i)}{\left[j_k^2 - \left(\frac{m_i a_k}{a_i}\right)^2 \right]}$$

$$= -\frac{4}{\pi}(-)^{\frac{n+m-1}{2}} \cdot \frac{n}{n^2 - m^2} \quad \text{for} \begin{cases} i = k \\ j_k = n \\ m_i = m \end{cases}$$

$$u_{o_2}^{(k)} \equiv \frac{1}{2a_2}\int_{-a_2}^{a_2} \cosh \lambda_k^! x \, dx = \frac{\sinh(\lambda_k^! a_2)}{(\lambda_k^! a_2)}$$

$$u_{n_2}^{(k)} \equiv \frac{1}{a_2}\int_{-a_2}^{a_2} \cosh \lambda_k^! x \cos n_2^! x \, dx = \frac{8}{\pi^2} \cdot (\lambda_k^! a_2) \cdot (-)^{\frac{n_2}{2}} \cdot \frac{\sinh(\lambda_k^! a_2)}{\left[n_2^2 + \left(\frac{\lambda_k a_2}{a_k}\right)^2\right]}$$

$$v_{m_2}^{(k)} \equiv \frac{1}{a_2}\int_{-a_2}^{a_2} \sinh \lambda_k^! x \sin m_2^! x \, dx = \frac{8}{\pi^2} \cdot (\lambda_k^! a_2) \cdot (-)^{\frac{m-1}{2}} \cdot \frac{\cosh(\lambda_k^! a_2)}{\left[m_2^2 + \left(\frac{\lambda_k a_2}{a_k}\right)^2\right]}$$

We write out the linear equations coming from the v boundary conditions then those arising from the conditions on Z.

$$\left\{\begin{aligned}
&(B_o^{1}\cos 2K_o^{1} l_1 + i\,A_o^{1}\sin 2K_o^{1} l_1)\, p_{o_1}^{1} = \theta_o \\
&(B_o^{1}\cos 2K_o^{1} l_1 + i\,A_o^{1}\sin 2K_o^{1} l_1)\, p_{n_1}^{1} + B_{n_1}^{1} = \theta_{n_1} \qquad n_1 = 2,4,6\ldots \\
&(A_o^{1}\cos 2K_o^{1} l_1 + i\,B_o^{1}\sin 2K_o^{1} l_1)\, q_{m_1}^{1} - D_{m_1}^{1} = \theta_{m_1} \qquad m_1 = 1,3,5\ldots
\end{aligned}\right\} \tag{42}$$

$$\left\{\begin{aligned}
&\frac{h_1}{h_2}\cdot B_o^{1} p_{o_1}^{1} = (B_o^{2}\cos 2K_o^{1} l_2 + iA_o^{2}\sin 2K_o^{2} l_2)\, r_{o_1}^{2} \\
&\frac{h_1}{h_2}\{B_o^{1} p_{n_1}^{1} - A_{n_1}^{1}\} = (B_o^{2}\cos 2K_o^{2} l_2 + iA_o^{2}\sin 2K_o^{2} l_2)\, r_{n_1}^{2} + \frac{a_2}{a_1}\cdot\sum_{n_2=2}^{\infty} P_{n_1 n_2} B_{n_2}^{2} \\
&\frac{h_1}{h_2}\{A_o^{1} q_{m_1}^{1} + C_{m_1}^{1}\} = (A_o^{2}\cos 2K_o^{2} l_2 + iB_o^{2}\sin 2K_o^{2} l_2)\, t_{m_1}^{2} - \frac{a_2}{a_1}\cdot\sum_{m_2=1}^{\infty} Q_{m_1 m_2} D_{m_2}^{2}
\end{aligned}\right\} \tag{43}$$

$$\left\{\begin{aligned}
&\frac{h_3}{h_2}\{(B_o^{3}\cos 2K_o^{3} l_3 + iA_o^{3}\sin 2K_o^{3} l_3)\, p_{o_3}^{3}\} = B_o^{2} r_{o_3}^{2} \\
&\frac{h_3}{h_2}\{(B_o^{3}\cos 2K_o^{3} l_3 + iA_o^{3}\sin 2K_o^{3} l_3)\, p_{n_3}^{3} + B_{n_3}^{3}\} = B_o^{2} r_{n_3}^{2} - \frac{a_2}{a_3}\sum_{n_3=2}^{\infty} P_{n_3 n_2} A_{n_2}^{2} \\
&\frac{h_3}{h_2}\{(A_o^{3}\cos 2K_o^{3} l_3 + iB_o^{3}\sin 2K_o^{3} l_3)\, q_{m_3}^{3} - D_{m_3}^{3}\} = A_o^{2} t_{m_3}^{2} + \frac{a_2}{a_3}\sum_{m_2=1}^{\infty} Q_{m_3 m_2} C_{m_2}^{2}
\end{aligned}\right\} \tag{44}$$

$$\left\{\begin{aligned}
&B_o^{3} = 0 \\
&A_{n_3}^{3} = 0 \\
&C_{m_3}^{3} = -A_o^{3} q_{m_3}^{3}
\end{aligned}\right\} \tag{45}$$

$$\left.\begin{aligned}
&\left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_o^{(1)} u_{o_2}^{(1)} - 2\omega_1\alpha_1 \sum_{m_1=1}^{\infty} \frac{P_{m_1 o}}{m_1} C_{m_1}^{(1)} + i\sigma\alpha_1 \sum_{n_1=2}^{\infty} \frac{P_{n_1 o}}{K_{n_1}} A_{n_1}^{(1)} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[\left(A_o^{(2)} \cos 2K_o^{(2)} l_2 + iB_o^{(2)} \sin 2K_o^{(2)} l_2\right) p_o^{(2)}\right] + 2\omega_2\alpha_2 \sum_{m_2=1}^{\infty} \frac{P_{m_2 o}}{m_2} D_{m_2}^{(2)} \\
&\left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_o^{(1)} u_{n_1}^{(1)} + i\sigma\alpha_1 \sum_{n_1=2}^{\infty} \frac{P_{n_1 n_2}}{K_{n_1}} A_{n_1}^{(1)} - 2\omega_1\alpha_1 \sum_{m_1=1}^{\infty} \frac{P_{m_1 n_2}}{m_1} C_{m_1}^{(1)} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[\left(A_o^{(2)} \cos 2K_o^{(2)} l_2 + iB_o^{(2)} \sin 2K_o^{(2)} l_2\right) p_{n_2}^{(2)}\right] + i\sigma\alpha_2 \frac{B_{n_2}}{K_{n_2}} + 2\omega_2\alpha_2 \sum_{m_2=1}^{\infty} \frac{P_{m_2 n_2}}{m_2} D_{m_2}^{(2)} \\
&\left(\frac{h_1}{g}\right)^{\frac{1}{2}} B_o^{(1)} v_{m_2}^{(1)} - i\sigma\alpha_1 \sum_{m_1=1}^{\infty} \frac{Q_{m_1 m_2}}{K_{m_1}} C_{m_1}^{(1)} - 2\omega_1\alpha_1 \sum_{n_1=2}^{\infty} \frac{Q_{n_1 m_2}}{n_1} A_{n_1}^{(1)} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[\left(B_o^{(2)} \cos 2K_o^{(2)} l_2 + iA_o^{(2)} \sin 2K_o^{(2)} l_2\right) q_{m_2}^{(2)}\right] - \frac{i\sigma\alpha_2}{K_{m_2}} D_{m_2}^{(2)} + 2\omega_2\alpha_2 \sum_{n_2=2}^{\infty} \frac{Q_{n_2 m_2}}{n_2} B_{n_2}^{(2)}
\end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned}
&\left(\frac{h_3}{g}\right)^{\frac{1}{2}} \left(A_o^{(3)} \cos 2K_o^{(3)} l_3 + iB_o^{(3)} \sin 2K_o^{(3)} l_3\right) u_{o_2}^{(3)} + i\sigma\alpha_3 \sum_{n_3=2}^{\infty} \frac{P_{n_3 o}}{K_{n_3}} B_{n_3}^{(3)} + 2\omega_3\alpha_3 \sum_{m_3=1}^{\infty} \frac{P_{m_3 o}}{m_3} D_{m_3}^{(3)} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_o^{(2)} p_{o_2}^{(2)} - 2\omega_2\alpha_2 \sum_{m_2=1}^{\infty} \frac{P_{m_2 o}}{m_2} C_{m_2}^{(2)} \\
&\left(\frac{h_3}{g}\right)^{\frac{1}{2}} \left(A_o^{(3)} \cos 2K_o^{(3)} l_3 + iB_o^{(3)} \sin 2K_o^{(3)} l_3\right) u_{n_2}^{(3)} + i\sigma\alpha_3 \sum_{n_3=2}^{\infty} \frac{P_{n_3 n_2}}{K_{n_3}} B_{n_3}^{(3)} + 2\omega_3\alpha_3 \sum_{m_3=1}^{\infty} \frac{P_{m_3 n_2}}{m_3} D_{m_3}^{(3)} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_o^{(2)} p_{n_2}^{(2)} + i\sigma\alpha_2 \frac{A_{n_2}^{(2)}}{K_{n_2}} - 2\omega_2\alpha_2 \sum_{m_2=1}^{\infty} \frac{P_{m_2 n_2}}{m_2} C_{m_2}^{(2)} \\
&\left(\frac{h_3}{g}\right)^{\frac{1}{2}} \left(B_o^{(3)} \cos 2K_o^{(3)} l_3 + iA_o^{(3)} \sin 2K_o^{(3)} l_3\right) v_{m_2}^{(3)} - i\sigma\alpha_3 \sum_{m_3=1}^{\infty} \frac{Q_{m_3 m_2}}{K_{m_3}} D_{m_3}^{(3)} + 2\omega_3\alpha_3 \sum_{n_3=2}^{\infty} \frac{Q_{n_3 m_2}}{n_3} B_{n_3}^{(3)} \\
&\quad = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} B_o^{(2)} q_{m_2}^{(2)} - i\sigma\alpha_2 \frac{C_{m_2}^{(2)}}{K_{m_2}} - 2\omega_2\alpha_2 \sum_{n_2=2}^{\infty} \frac{Q_{n_2 m_2}}{n_2} A_{n_2}^{(2)}
\end{aligned}\right\} \tag{47}$$

The Solution of the Linear Equations

Equations (42) to (47) represent 6x ∞ linear relations between 6x ∞ unknowns, all of these complex.

Equations (42) and (45) show that $A_m^{③}$, $C_m^{③}$, $B_n^{①}$, $D_m^{①}$ are really not involved in the system and may be evaluated once the Kelvin waves are known. A twofold infinity of unknowns is already eliminated.

By the way this makes the theorem on v more obvious since it can be seen that θ_n, and θ_m, the higher harmonics of the currents, are not involved in the equations giving the arbitrary constants in 2. Only $B_n^{①}$, and $D_m^{①}$, which describe the current in the vicinity of the mouth depend on these quantities.

We use (43) and (44) to express the arbitrary constants in 1 and 3 in terms of sea 2.

We end up with a system of 2x ∞ linear relations between 2x ∞ unknowns:

$$\begin{pmatrix} B_o^{(2)} \\ B_o^{(2)}\cos 2K_o^{(2)} l_2 + i A_o^{(2)} \sin 2K_o^{(2)} l_2 \end{pmatrix} (-)^{\frac{k-1}{2}}\, i \Bigg[\frac{\sigma\alpha_k h_2}{h_k} \sum_{n_k=2}^{\infty} \frac{P_{n_k n}}{K_{n_k}} \cdot r_{n_k}^{(2)} - G^{(k)} \Big\{ \operatorname{cotg}(2K_o^{(k)} l_k) \Big[\Big(\frac{h_k}{g}\Big)^{\frac{1}{2}} u_n^{(k)}$$

$$+ \sigma\alpha_k \sum_{n_k=2}^{\infty} \frac{P_{n_k n}}{K_{n_k}}\, p_{n_k}^{(k)} \Big\} \Big] + 2\omega_k\alpha_k \sum_{m_k=1}^{\infty} \frac{P_{n_k n}}{m_k}\, q_{n_k}^{(k)} \Bigg]$$

$$+ \begin{pmatrix} A_o^{(2)} \\ A_o^{(2)}\cos 2K_o^{(2)} l_2 + i B_o^{(2)} \sin 2K_o^{(2)} l_2 \end{pmatrix} \Bigg[\Big(\frac{h_2}{g}\Big)^{\frac{1}{2}} p_n^{(2)} + \frac{2\omega_k\alpha_k\, h_2}{h_k} \sum_{m_k=1}^{\infty} \frac{P_{m_k n}}{m_k}\, t_{m_k}^{(2)} \Bigg]$$

$$+ \frac{\sigma\alpha_2}{K_n} \begin{pmatrix} A_n \\ B_n \end{pmatrix} \pm \sigma\alpha_k \frac{h_2 a_2}{h_k a_k} \sum_{N=2}^{\infty} \begin{pmatrix} A_n \\ B_n \end{pmatrix} \sum_{n_k=2}^{\infty} \frac{P_{n_k n}\, P_{n_k N}}{K_{n_k}}$$

$$- \sum_{m=1}^{\infty} \begin{pmatrix} C_m \\ -D_m \end{pmatrix} \Bigg[2\omega_2\alpha_2 \frac{\dot{P}_{mn}}{m} - 2\omega_k\alpha_k \frac{h_2 a_2}{h_k a_k} \sum_{m_k=1}^{\infty} \frac{P_{m_k n}\, Q_{m_k m}}{m_k} \Bigg]$$

$$= \begin{pmatrix} 0 \\ \dfrac{F^{(1)}}{i \sin 2K_o^{(1)} l_1} \Bigg[\Big(\dfrac{h_1}{g}\Big)^{\frac{1}{2}} u_n^{(1)} + 2\omega_1\alpha_1 \displaystyle\sum_{m_1=1}^{\infty} \dfrac{P_{m_1 n_m}}{m_1}\, q_{m_1}^{(1)} \end{pmatrix} \cdot \qquad n = 0,2,4,6\ldots.$$

$$\cdot\begin{pmatrix} B_o^{(2)} \\ B_o^{(2)}\cos 2K_o^{(2)} l_2 + i A_o^{(2)} \sin 2K_o^{(2)} l_2 \end{pmatrix} \left[\left(\frac{h_2}{g}\right)^{\frac{1}{2}} q_m^{(2)} + G^{(k)} \left\{ -\left(\frac{h_k}{g}\right)^{\frac{1}{2}} v_m^{(k)} + \sigma \alpha_k \operatorname{cotg}(2K_o^{(k)} l_k) \cdot \sum_{m_k=1}^{\infty} \frac{Q_{m_k m}}{K_{m_k}} q_{m_k}^{(k)} \right.\right.$$

$$\left.\left. + 2\omega_k \alpha_k \sum_{n_k=2}^{\infty} \frac{Q_{n_k m}}{n_k} \cdot p_{n_k}^{(k)} \right\} - 2\omega_k \alpha_k \frac{h_2}{h_k} \sum_{n_k=2}^{\infty} \frac{Q_{n_k m}}{n_k} r_{n_k}^{(2)} \right]$$

$$+ (-)^{\frac{k-1}{2}} \begin{pmatrix} A_o^{(2)} \\ A_o^{(2)}\cos 2K_o^{(2)} l_2 + i B_o^{(2)} \sin 2K_o^{(2)} l_2 \end{pmatrix} i\sigma\alpha_k \frac{h_2}{h_k} \sum_{m_k=1}^{\infty} \frac{Q_{m_k m}}{K_{m_k}} t_{n_k}^{(2)}$$

$$- i\sigma \left[\frac{\alpha_2}{K_m} \begin{pmatrix} C_m \\ D_m \end{pmatrix} + \alpha_k \frac{h_2 a_2}{h_k a_k} \sum_{M=1}^{\infty} \begin{pmatrix} C_M \\ D_M \end{pmatrix} \sum_{m_k=1}^{\infty} \frac{Q_{m_k M} Q_{m_k m}}{K_{m_k}} \right]$$

$$+ i \sum_{n=2}^{\infty} \begin{pmatrix} A_n \\ B_n \end{pmatrix} \left[2\omega_2 \alpha_2 \frac{Q_{nm}}{n} - 2\omega_k \alpha_k \frac{h_2 a_2}{h_k a_k} \sum_{n_k=2}^{\infty} \frac{P_{n_k n} Q_{n_k m}}{n_k} \right]$$

$$= \begin{pmatrix} 0 \\ \dfrac{\sigma\alpha_1 F^{(1)}}{\sin 2K_o^{(1)} l_1} \sum_{m_1=1}^{\infty} \dfrac{Q_{m_1 m}}{K_{m_1}} q_{m_1}^{(1)} \end{pmatrix} \qquad m = 1,3,5\dots$$

$$F^{(1)} \equiv \frac{\theta_o}{p_o^{(1)}} \qquad G^{(1)} \equiv \frac{r_{o_1}^{(2)}}{p_o^{(1)}} \cdot \frac{h_2}{h_1} \qquad G^{(3)} \equiv \frac{r_{o_3}^{(2)}}{p_o^{(3)}} \cdot \frac{h_2}{h_3}$$

k stands for 1 or 3. The index 2 has been dropped and the equation for $n = 0$ is contained in the above set if we are ready to accept that A_o is equal to 0 in the Poincaré part of the expression. We have redefined

$$i\begin{pmatrix} A_n \\ B_n \end{pmatrix} \to \begin{pmatrix} A_n \\ B_n \end{pmatrix} \qquad n \neq o$$

With this definition, if we break up system (48) into its real and imaginary parts we obtain two independent systems of equations in the unknowns

$$\left[A_o^R, B_o^I, \begin{pmatrix} A_n \\ B_n \end{pmatrix}^R, \begin{pmatrix} C_m \\ D_m \end{pmatrix}^R\right] \quad \text{expressed in terms of} \quad \theta_o^I$$

$$\left[A_o^I, B_o^R, \begin{pmatrix} A_n \\ B_n \end{pmatrix}^I, \begin{pmatrix} C_m \\ D_m \end{pmatrix}^I\right] \quad \text{expressed in terms of} \quad \theta_o^R$$

where R and I denote the real or imaginary part of the quantity. Were it not of the assumption of complete reflection at $+ 1_3$ all these quantities would have been interrelated.

If θ_o^I or θ_o^R happens to vanish (i. e. if the time origin is suitably chosen) one set of unknowns will vanish as well.

In either set of equations there is $1x\ \infty$ relations between A_o, B_o and A_n, C_m and another infinity between A_o, B_o and B_n, D_m. The two systems of equations would be independent would it not be of the simultaneous presence of A_o and B_o in both sets.

We use this feature by moving B_o to the right and add it to the column vector involving θ_o. In this way we have two systems of equations in the unknowns (A_o, A_n, C_m) and (A_o, B_n, D_m) which we write in matrix notation as

$$\begin{Bmatrix} \Gamma + \times A_+ = \theta_o\,\delta_+ + B_o\epsilon_+ \\ \Gamma + \times A_- = \theta_o\delta_- + B_o\epsilon_- \end{Bmatrix} \qquad (49)$$

where

$$A_+ \equiv \begin{pmatrix} A_o \\ A_n \\ C_m \end{pmatrix} \qquad A_- \equiv \begin{pmatrix} A_o \\ B_n \\ D_m \end{pmatrix}$$

$\delta_{\pm}$; $\epsilon_{\pm}$ are the column vectors of the coefficients of θ_0 and B_0 in the two systems

$\Gamma_{\pm}$ is the matrix of the coefficients of $A_{\pm}$

The solution of (49) is

$$A_{\pm} = \theta_o \Gamma_{\pm}^{-1} \times \delta_{\pm} + B_o \Gamma_{\pm}^{-1} \times \epsilon_{\pm} \qquad (50)$$

and we know

$$(A_+)_o = (A_-)_o$$

so that

$$\theta_o (\Gamma_+^{-1} \times \delta_+)_o + B_o(\Gamma_+^{-1} \times \epsilon_+)_o = \theta_o(\Gamma_-^{-1} \times \delta_-) + B_o(\Gamma_-^{-1} \times \epsilon_-)_o$$

or

$$B_o = \frac{(\Gamma_-^{-1} \times \delta_-)_o - (\Gamma_+^{-1} \times \delta_+)_o}{(\Gamma_+^{-1} \times \epsilon_+)_o - (\Gamma_-^{-1} \times \epsilon_-)_o} \cdot \theta_o \qquad (51)$$

The last relation gives an explicit value for B_0 which is substituted into the column vectors $\theta_o \delta_{\pm} + B_o \epsilon_{\pm}$

Using $\Gamma_{\pm}^{-1}$ we obtain values for $A_{\pm}$ in (50).

The two systems of equations are quasiindependent because we have assumed the sea to be elongated. We have noted previously that this assumption does not hold well for M_2 and $j = 1$; there, rather than the unknowns C_1 and D_1 we should have written

$$C_1 \pm e^{-2 l_2} D_1, \; C_1 e^{-2 l_2} \pm D_1$$

Using the same method as above we express (48) as

$$\Gamma_+ \times A_+ = \theta_o \delta_+ + B_o \epsilon_+ + e^{-2l_2} D_1 \zeta_+$$
$$\Gamma_- \times A_- = \theta_o \delta_- + B_o \epsilon_- + e^{-2l_2} C_1 \zeta_- \qquad (52)$$

We know
$$(A_\pm)_o = A_o$$
$$(A_+)_1 = C_1$$
$$(A_-)_1 = D_1$$

which results in the system of three relations between the three unknowns B_o, C_1 and D_1

$$\theta_o(\Gamma_+^{-1} \times \delta_+)_o + B_o(\Gamma_+^{-1} \times \epsilon_+)_o + e^{-2l_2} D_1(\Gamma_+^{-1} \times \zeta_+)_o = \theta_o(\Gamma_-^{-1} \times \delta_-)_o + B_o(\Gamma_-^{-1} \times \epsilon_-)_o + e^{-2l_2} C_1(\Gamma_-^{-1} \times \zeta_-)_o$$
$$C_1 = \theta_o(\Gamma_+^{-1} \times \delta_+)_1 + B_o(\Gamma_+^{-1} \times \epsilon_+)_1 + e^{-2l_2} D_1(\Gamma_+^{-1} \times \zeta_+)_1$$
$$D_1 = \theta_o(\Gamma_-^{-1} \times \delta)_1 + B_o(\Gamma_-^{-1} \times \epsilon_-)_1 + e^{-2l_2} C_1(\Gamma_-^{-1} \times \zeta_-)_1 \qquad (53)$$

from which the column vectors can be evaluated and $A\pm$ as well.

Thus by finding the value of the two inverse matrices

$$\Gamma_+^{-1} \quad , \quad \Gamma_-^{-1}$$

we may find four times as many unknowns as the rank of the matrix. $\Gamma_\pm$ for the real and complex parts happen to be identical, and so, if we choose to solve for 40 complex constants in (50), these being determined by two independent sets of 40 linear equations, all we need is to invert two 20 x 20 matrices.

Once these are inverted it is an easy task to find B_o, then the remaining unknowns in 2. From the extra v relations at $-l_1$, and $+l_3$, we get the constants for basins 1 and 3. The number of unknown complex constant determined in this way is 120 which is equivalent to 240 unknown real constants.

Balancing the Kelvin Waves Only

Before we proceed writing out the matrices and the column vectors arising from the previous equations, it is good to pause a moment to verify that the boundary conditions (32) to (37) contain to a zero order of approximation those connecting three channels of constant width and depth.

We have seen previously that the Kelvin waves are the two dimensional equivalent of the sinusoidal waves that appear in a channel of constant width and depth when the motion is purely linear; we now wish to show that boundary conditions (32) to (37) will coalesce into those relating 3 channels of constant width and depth when two dimensional motion is neglected.

The solution is a system of three channels of constant depth h_j, length l_j and width $2a_j$ assuming perfect reflection at the head, is given by

$$v_1 = B_1 \cos K_1(y-l_1) - A_1 \sin K_1(y-l_1) \quad Z_1 = \left(\frac{h_1}{g}\right)^{\frac{1}{2}} \left[-B_1 \sin K_1(y-l_1) - A_1 \cos K_1(y-l_1)\right]$$

$$v_2 = B_2 \cos K_2(y-l_2) - A_2 \sin K_2(y-l_2) \quad Z_2 = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[-B_2 \sin K_2(y-l_2) - A_2 \cos K_2(y-l_2)\right] \qquad (54)$$

$$v_3 = -A_3 \sin K_3(y-l_3) \quad Z_3 = -\left(\frac{h_3}{g}\right)^{\frac{1}{2}} A_3 \cos K_3(y-l_3)$$

The boundary conditions are

$$\begin{aligned}
v_1(-l_1) &= \theta_0 \\
h_1 a_1 v_1(+l_1) &= h_2 a_2 v_2(-l_2) \\
Z_1(+l_1) &= Z_2(-l_2) \\
h_2 a_2 v_2(+l_2) &= h_3 a_3 v_3(-l_3) \\
Z_2(+l_2) &= Z_3(-l_3) \\
v_3(+l_3) &= 0 \qquad (55)
\end{aligned}
\quad \longleftrightarrow \quad
\begin{aligned}
A_1 \sin 2K_1 l_1 + B_1 \cos 2K_1 l_1 &= \theta_0 \\
h_1 a_1 B_1 &= h_2 a_2 (A_2 \sin 2K_2 l_2 + B_2 \cos 2K_2 l_2) \\
\left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_1 &= \left(\frac{h_2}{g}\right)^{\frac{1}{2}} (A_2 \cos 2K_2 l_2 - B_2 \sin 2K_2 l_2) \\
h_2 a_2 B_2 &= h_3 a_3 A_3 \sin 2K_3 l_3 \\
\left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_2 &= \left(\frac{h_3}{g}\right)^{\frac{1}{2}} A_3 \cos 2K_3 l_3 \qquad (56)
\end{aligned}$$

If in (42) to (47), we balance the Kelvin waves only we have

$$B_o^{(1)} \cos 2K_1 l_1 + i A_o^{(1)} \sin 2K_1 l_1 = \frac{\theta_o}{p_{o_1}^{(1)}}$$

$$\frac{h_1}{h_2} B_o^{(1)} p_{o_1}^{(1)} = (B_o^{(1)} \cos 2K_2 l_2 + i A_o^{(2)} \sin 2K_2 l_2)\; r_{o_1}^{(2)}$$

$$\left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_o^{(1)} u_{o_2}^{(2)} \simeq \left(\frac{h_2}{g}\right)^{\frac{1}{2}} \left[A_o^{(2)} \cos 2K_2 l_2 + i B_o^{(2)} \sin 2K_2 l_2\right] p_o^{(2)} \tag{57}$$

$$\frac{h_3}{h_2}\left[B_o^{(3)} \cos 2K_3 l_3 + i A_o^{(3)} \sin 2K_3 l_3\right] p_o^{(3)} = B_o^{(2)} r_{o_3}^{(2)}$$

$$\left(\frac{h_3}{g}\right)^{\frac{1}{2}} \left[A_o^{(3)} \cos 2K_3 l_3 + i B_o^{(3)} \sin 2K_3 l_3\right] u_{o_2}^{(3)} \simeq \left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_o^{(2)} p_o^{(2)}$$

$$B_o^{(3)} = 0$$

We notice that as $\lambda' \to 0$

$$\frac{1}{p_{o_1}^{(1)}} \equiv \frac{\lambda_1 a_1}{\sin h \lambda_1' a_1} \to 1$$

$$\frac{p_o^{(2)}}{u_{o_2}^{(1)}} \equiv \frac{\sin h\, \lambda_2' a_2}{\lambda_2' a_2} \cdot \frac{\lambda_1' a_2}{\sin h\, \lambda_1' a_2} \to 1 \qquad \frac{p_o^{(2)}}{u_{o_2}^{(3)}} \equiv \frac{\sin h\, \lambda_2' a_2}{\lambda_2' a_2} \cdot \frac{\lambda_3' a_2}{\sin h\, \lambda_3' a_2} \to 1$$

$$\frac{r_{o_1}^{(2)}}{p_{o_1}^{(1)}} \equiv \frac{h_2}{h_1} \cdot \frac{\sin h \lambda_2' a_2}{\lambda_2' a_1} \cdot \frac{\lambda_1' a_1}{\sin h(\lambda_1' a_1)} \to \frac{h_2}{h_1}\; \frac{a_2}{a_1} \qquad \frac{h_2}{h_3}\; \frac{r_{o_3}^{(2)}}{p_o^{(3)}} \equiv \frac{h_2}{h_3} \cdot \frac{\sin h\, \lambda_2' a_2}{\lambda_2' a_3} \cdot \frac{\lambda_3' a_3}{\sin h\, \lambda_3' a_3} \to \frac{h_2}{h_3}\; \frac{a_2}{a_3}$$

In this limit (57) becomes

$$
\begin{aligned}
& i A_o^{1} \sin 2K_1 l_1 + B_o^{1} \cos 2K_1 l_1 = \theta_o \\
& h_1 a_1 B_o^{1} = h_2 a_2 [B_o^{2} \cos 2K_2 l_2 + i A_o^{2} \sin 2K_2 l_2] \\
& \left(\frac{h_1}{g}\right)^{\frac{1}{2}} A_o^{1} = \left(\frac{h_2}{g}\right)^{\frac{1}{2}} [A_o^{2} \cos 2K_2 l_2 + i B_o^{2} \sin 2K_2 l_2] \\
& h_2 a_2 B_o^{2} = h_3 a_3 i A_o^{3} \sin 2K_3 l_3 \\
& \left(\frac{h_2}{g}\right)^{\frac{1}{2}} A_o^{2} = \left(\frac{h_3}{g}\right)^{\frac{1}{2}} A_o^{3} \cos 2K_3 l_3
\end{aligned}
\tag{58}
$$

The two systems of equations (58) and (56) have identical coefficients and the same column vector and therefore we conclude

$$
i A_o^{j} \equiv A_j \qquad B_o^{j} \equiv B_j
$$

i.e. balancing the Kelvin waves only is equivalent to a zero order of approximation to balancing the sinusoidal waves in a system of channels of constant width and depth.

Expressions for the Matrices and the Column Vectors

$$\Gamma_+$$

$$
\begin{array}{c}
 \\
\text{even} \\
 \\
\text{odd}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\quad 0 \quad & \qquad\qquad \text{odd} \qquad\qquad & \qquad \text{even}
\end{array} \\
\left(
\begin{array}{ccc}
\left(\frac{h_2}{g}\right)^{\frac{1}{2}} \overset{(2)}{p}_n + 2\omega_3\alpha_3 \frac{h_2}{h_3} \cdot \sum_{m_3=1}^{\infty} \frac{P_{m_3 n}}{m} \overset{(2)}{t}_{m_3}, & -2\omega_2\alpha_2 \frac{P_{mn}}{m} + 2\omega_3\alpha_3 \frac{h_2 a_2}{h_3 a_3} \cdot \sum_{m_3=1}^{\infty} \frac{P_{m_3 n}\, Q_{m_3 n}}{m_3}, & \frac{\sigma\alpha_2}{K_n} \delta_{nN} + \sigma\alpha_3 \frac{h_2 a_2}{h_3 a_3} \cdot \sum_{m_3=1}^{\infty} \frac{n_3 n \cdot P_{n_3 N}}{K_{n_3}} \\
 & & \\
-\sigma\alpha_3 \frac{h_2}{h_3} \sum_{m_3=1}^{\infty} \frac{Q_{m_3 m}}{K_{m_3}} \overset{(2)}{t}_{m_3}, & -\frac{\sigma\alpha_2}{K_m} \delta_{mM} - \sigma\alpha_3 \frac{h_2 a_2}{h_3 a_3} \cdot \sum_{m_3=1}^{\infty} \frac{Q_{m_3 M}\, Q_{m_3 m}}{K_{m_3}}, & \frac{2\omega_2\alpha_2 Q_{nm}}{n} - 2\omega_3\alpha_3 \frac{h_2 a_2}{h_3 a_3} \sum_{n_3=2}^{\infty} \frac{P_{n_3 n}\, Q_{n_3 m}}{n_3}
\end{array}
\right)
\end{array}
$$

$$\Gamma_-$$

$$\begin{array}{ccc} & 0 \quad \text{odd} & \text{even} \\ \begin{array}{c}\text{even}\\[4em]\text{odd}\end{array} & \multicolumn{2}{c}{\left(\begin{array}{ll} \Gamma_{-\,\text{even},0}\,, & 2\omega_2\alpha_2 \dfrac{P_{mn}}{m} - 2\omega_1\alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \displaystyle\sum_{m_1=1}^{\infty} \dfrac{P_{m_1n}}{m_1} \cdot Q_{m_1m}\,, \quad \dfrac{\sigma\alpha_2}{K_n}\,\delta_{nN} + \alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \displaystyle\sum_{n_1=2}^{\infty} \dfrac{P_{n_1n}P_{n_1N}}{K_{n_1}} \\[3em] \Gamma_{-\,\text{odd},0} & , -\dfrac{\sigma\alpha_2}{K_m}\,\delta_{mM} - \sigma\alpha_1 \dfrac{h_2a_2}{h_1a_1} \displaystyle\sum_{m_1=1}^{\infty} \dfrac{Q_{m_1M}\,Q_{m_1m}}{K_{m_1}}\,, \quad -2\omega_2\alpha_2 \dfrac{Q_{nm}}{n} + 2\omega_1\alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \displaystyle\sum_{n_1=2}^{\infty} \dfrac{P_{n_1n}\,Q_{n_1m}}{n_1} \end{array}\right)} \end{array}$$

$$\Gamma_{-\,\text{even},0} \equiv \cos 2K_o^{(2)} l_2 \left[\left(\frac{h_2}{g}\right)^{\frac{1}{2}} p_n^{(2)} + 2\omega_1\alpha_1 \frac{h_2}{h_1} \sum_{m_1=1}^{\infty} \frac{P_{m_1n}}{m_1}\, t_{m_1}^{(2)}\right] - \sin 2K_o^{(2)} l_2 \left[\frac{\sigma\alpha_1 h_2}{h_1} \sum_{n_1=2}^{\infty} \frac{P_{n_1n}}{K_{n_1}}\, r_{n_1}^{(2)} - G^{(1)} \left\{\operatorname{cotg} 2K_o^{(1)} l_1 \left(\left(\frac{h_1}{g}\right)^{\frac{1}{2}} u_n^{(1)} + 2\omega_1\alpha_1 \sum_{m_1=1}^{\infty} \frac{P_{m_1n}}{m_1}\, q_{m_1}^{(1)}\right) + \sigma\alpha_1 \sum_{n_1=2}^{\infty} \frac{P_{n_1n_2}}{K_{n_2}}\, p_{n_1}^{(1)}\right\}\right]$$

$$\Gamma_{-\,\text{odd},0} \equiv \sin 2K_o^{(1)} l_2 \left[\left(\frac{h_2}{g}\right)^{\frac{1}{2}} q_{m_2}^{(2)} + G^{(1)} \left\{-\left(\frac{h_1}{g}\right)^{\frac{1}{2}} v_m^{(1)} + \sigma\alpha_1 \operatorname{cotg} 2k_o^{(1)} l_{3_2} \cdot \sum_{m_1=1}^{\infty} \frac{Q_{m_1m}}{K_{m_1}}\, q_{m_1}^{(1)} + 2\omega_1\alpha_1 \sum_{n_1=1}^{\infty} \frac{Q_{n_1m}}{n_1}\, p_{n_1}^{(1)}\right\} - 2\omega_1\alpha_1 \frac{h_2}{h_1} \cdot \sum_{n_1=2}^{\infty} \frac{Q_{n_1m}}{n_1}\, r_{n_1}^{(2)}\right] + \cos 2K_o^{(2)} l_2 \left[\sigma\alpha_1 \frac{h_2}{h_1} \cdot \sum_{m_1=1}^{\infty} \frac{Q_{m_1m}}{K_{m_1}} \cdot t_{m_1}^{(2)}\right]$$

$$\zeta_+ = \left(\begin{array}{l} -2\omega_2\alpha_2 P_{1n} + 2\omega_3\alpha_3 \dfrac{h_2a_2}{h_3a_3} \displaystyle\sum_{m_3=1}^{\infty} \dfrac{P_{m_3 n}\, Q_{m_3 1}}{m_3} \\[2em] \dfrac{\sigma\alpha_2\delta_{1m}}{K_1} - \sigma\alpha_3 \dfrac{h_2a_2}{h_3a_3} \cdot \displaystyle\sum_{m_3=1}^{\infty} \dfrac{Q_{m_3 m}\, Q_{m_3 1}}{K_{m_3}} \end{array} \right)$$

$$\zeta_- = \left(\begin{array}{l} 2\omega_2\alpha_2 P_{1n} - 2\omega_1\alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \displaystyle\sum_{m_1=1}^{\infty} \dfrac{P_{m_1 n}\, Q_{m_1 1}}{m_1} \\[2em] \dfrac{\sigma\alpha_2}{K_1}\delta_{1n} - \sigma\alpha_1 \dfrac{h_2a_2}{h_1a_1} \cdot \displaystyle\sum_{m_1=1}^{\infty} \dfrac{Q_{m_1 m}\, Q_{m_1}}{K_{m_1}} \end{array} \right)$$

$$\delta_+ = 0$$

$$\delta_- = \left(\begin{array}{l} -\dfrac{iF^{1}}{\sin 2K_o^{1} l_1} \left[\left(\dfrac{h_1}{g}\right)^{-\frac{1}{2}} u_n^{1} + 2\omega_1\alpha_1 \displaystyle\sum_{m_1=1}^{\infty} \dfrac{P_{m_1 n}}{m_1}\, q_{m_1}^{1} \right] \\[2em] -\dfrac{i\sigma\alpha_1 F^{1}}{\sin 2K_o^{1} l_1} \cdot \displaystyle\sum_{m_1=1}^{\infty} \dfrac{Q_{m_1 m}}{K_{m_1}}\, q_{m_1}^{1} \end{array} \right) \begin{array}{l} \text{even} \\[2em] \text{odd} \end{array}$$

$$\varepsilon_- = i \begin{pmatrix} \sigma\alpha_3 \frac{h_2}{h_3} \cdot \sum_{n_3=2}^{\infty} \frac{P_{3n}}{K_{n_3}} r_{n_3}^{(2)} - G^{(3)}\left[(\cot g 2K_o^{(3)} l_3)\left(\left(\frac{h_3}{g}\right)^{\frac{1}{2}} u_n^{(3)} + 2\omega_3\alpha_3 \sum_{m_3=1}^{\infty} \frac{P_{m_3 n}}{m_3} q_{m_3}^{(3)}\right) + \sigma\alpha_3 \sum_{n_3=2}^{\infty} \frac{P_{n_3 n}}{K_{n_3}} p_{n_3}^{(3)}\right] & \text{even} \\ \left(\frac{h_2}{g}\right)^{\frac{1}{2}} q_m^{(2)} + G^{(3)}\left[-\left(\frac{h_3}{g}\right)^{\frac{1}{2}} v_m^{(3)} + \sigma\alpha_3 \cot g 2K_o^{(3)} l_3 \cdot \sum_{m_3=1}^{\infty} \frac{Q_{m_3 m}}{K_{m_3}} q_{m_3}^{(3)} + 2\omega_3\alpha_3 \sum_{n_3=2}^{\infty} \frac{Q_{n_3 m}}{n_3} \cdot p_{n_3}^{(3)}\right] - 2\omega_3\alpha_3 \frac{h_2}{h_3} \sum_{n_3=2}^{\infty} \frac{Q_{n_3 m}}{n_3} r_{n_3}^{(2)} & \text{odd} \end{pmatrix}$$

$$\varepsilon_- = i \begin{bmatrix} -\cos 2K_o^{(2)} l_2 \left[\sigma\alpha_1 \frac{h_2}{h_1} \cdot \sum_{n_1=2}^{\infty} \frac{P_{n_1 n}}{K_{n_1}} r_{n_1}^{(2)} - G^{(1)}\left\{\cot g 2K_o^{(1)} l_1 \left(\frac{h_1}{g}\right)^{\frac{1}{2}} u_n^{(1)} + 2\omega_1\alpha_1 \sum_{m_1=1}^{\infty} \frac{P_{m_1 n}}{m_1} q_{m_1}^{(1)} + \sigma\alpha_1 \sum_{n_1=2}^{\infty} \frac{P_{n_1 n}}{K_{n_1}} p_{n_1}^{(1)}\right\}\right] & \\ -\sin 2K_o^{(2)} l_2 \left[\left(\frac{h_2}{g}\right)^{\frac{1}{2}} p_n^{(2)} + 2\omega_1\alpha_1 \frac{h_2}{h_1} \sum_{m_1=1}^{\infty} \frac{P_{m_1 n}}{m_1} t_{m_1}^{(2)}\right] & \text{even} \\ \cos 2K_o^{(2)} l_2 \left[\left(\frac{h_2}{g}\right)^{\frac{1}{2}} q_m^{(2)} + G^{(1)}\left\{-\left(\frac{h_1}{g}\right)^{\frac{1}{2}} v_m^{(1)} + \sigma\alpha_1 \cot g 2K_o^{(1)} l_1 \sum_{m_1=1}^{\infty} \frac{Q_{m_1 m}}{K_{m_1}} q_{m_1}^{(1)} + 2\omega_1\alpha_1 \sum_{n_1=2}^{\infty} \frac{Q_{n_1 m}}{n_1} p_{n_1}^{(1)}\right\} - 2\omega_1\alpha_1 \frac{h_2}{h_1} \sum_{n_1=2}^{\infty} \frac{Q_{n_1 m}}{n_1} r_{n_1}^{(2)}\right] & \\ -\sin 2K_o^{(2)} l_2 \cdot \sigma\alpha_1 \frac{h_2}{h_1} \sum_{m_1=1}^{\infty} \frac{Q_{m_1 m}}{K_m} t_{m_1}^{(2)} & \text{odd} \end{bmatrix}$$

The elements of the matrices and of the column vectors involve a set of infinite series which have first to be evaluated before we can proceed to the solution of the linear equations. Such infinite series are given by the formulas

$$\sum_{n_k=2}^{\infty} \frac{P_{n_k n}}{K_{n_k}} \; p_{n_k}^{(k)} = \frac{32}{\pi^3}\,\frac{a_k}{a_2} \cdot (\lambda'_k a_k) \cdot \sinh(\lambda'_k a_k)\,(-)^{\frac{n}{2}} \sum_{n_k=2}^{\infty} \frac{n_k}{K_{n_k}}\,\frac{(-)^{\frac{n_k}{2}}\sin(n'_k a_2)}{\left(n_k^2+\lambda_k^2\right)\left(n_k^2-\left(\frac{n a_k}{a_2}\right)^2\right)} \qquad (2)$$

$$\sum_{m_k=1}^{\infty} \frac{P_{m_k n}}{m_k} \; q_{m_k}^{(k)} \equiv \frac{32}{\pi^3} \cdot (\lambda' a_k) \cdot \cosh(\lambda'_k a_k)\,(-)^{\frac{n}{2}} \sum_{m_k=1}^{\infty} (-)^{\frac{m_k-1}{2}}\,\frac{\sin(m'_k a_2)}{\left(m_k^2+\lambda_k^2\right)\left(m_k^2-\left(\frac{n a_k}{a_2}\right)^2\right)} \qquad (1)$$

$$\sum_{n_k=2}^{\infty} \frac{Q_{n_k m}}{n_k} \; p_{n_k}^{(k)} \equiv -\frac{32}{\pi^3}\,\frac{a_k}{a_2} \cdot (\lambda'_k a_k) \cdot \sinh(\lambda'_2 a_k)\,(-)^{\frac{m-1}{2}} \sum_{n_k=2}^{\infty} \frac{(-)^{\frac{n_k}{2}}\cos(n'_k a_2)}{\left(n_k^2+\lambda_k^2\right)\left(n_k^2-\left(\frac{m a_k}{a_2}\right)^2\right)} \qquad (12)$$

$$\sum_{m_k=1}^{\infty} \frac{Q_{m_k m}}{K_{m_k}} \; q_{m_k}^{(k)} \equiv -\frac{32}{\pi^3}\,\frac{a_k}{a_2} \cdot (\lambda'_k a_k)\cosh(\lambda'_k a_k)\,(-)^{\frac{m-1}{2}} \sum_{m_k=1}^{\infty} \frac{m_k}{K_{m_n}}\,\frac{(-)^{\frac{m_k-1}{2}}\cos(m'_k a_2)}{\left(m_k^2+\lambda_k^2\right)\left(m_k^2-\left(\frac{m a_k}{a_2}\right)^2\right)} \qquad (11)$$

$$\sum_{n_k=2}^{\infty} \frac{P_{n_k n}}{K_{n_k}} \; r_{n_k}^{(2)} \equiv \frac{16}{\pi^2}\,\frac{a_k}{a_2}\,(-)^{\frac{n}{2}} \left\{ \frac{\lambda_2 a_k}{a_2} \sinh\lambda'_2 a_2 \cdot \sum_{n_k=2}^{\infty} \frac{n_k}{K_{n_k}} \cdot \frac{\sin(n'_k a_2)\cos(n'_k a_2)}{\left(n_k^2+\left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(n_k^2-\left(\frac{n a_k}{a_2}\right)^2\right)} \right. \qquad (4)$$

$$\left. + \cosh\lambda'_2 a_2 \sum_{n_k=2}^{\infty} \frac{n_k^2 \cdot \sin^2(n'_k a_2)}{K_{n_k}\left(n_k^2+\left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(n_k^2-\left(\frac{n a_k}{a_2}\right)^2\right)} \right\} \qquad (6)$$

$$\sum_{m_k=1}^{\infty} \frac{P_{m_k n}}{m_k} \; t_{m_k}^{(2)} \equiv \frac{16}{\pi^2}\,\frac{a_k}{a_2}\,(-)^{\frac{n}{2}} \left\{ \frac{\lambda_2 a_k}{a_2} \cosh\lambda'_2 a_2 \sum_{m_k=1}^{\infty} \frac{\sin^2(m'_k a_2)}{\left(m_k^2+\left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(m_k^2-\left(\frac{n a_k}{a_2}\right)^2\right)} \right. \qquad (5)$$

$$\left. - \sinh\lambda'_2 a_2 \sum_{m_k=2}^{\infty} \frac{m_k \sin(m'_k a_2)\cos(m'_k a_2)}{\left(m_k^2+\left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(m_k^2-\left(\frac{n a_k}{a_2}\right)^2\right)} \right\} \qquad (3)$$

$$\sum_{m_k=1}^{\infty}\frac{Q_{m_k m}}{K_{m_k}}\,t^{(2)}_{m_k} = -\frac{16}{\pi^2}\,\frac{a_k}{a_2}\,(-)^{\frac{m-1}{2}}\left\{\frac{\lambda_2 a_k}{a_2}\cosh\lambda_2' a_2\sum_{m_k=1}^{\infty}\frac{m_k}{K_{m_k}}\cdot\frac{\sin(m_k' a_2)\cos(m_k' a_2)}{\left(m_k^2+\left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(m_k^2-\left(\frac{m a_k}{a_2}\right)^2\right)}\right. \quad (6)$$

$$\left. -\sinh\lambda_2' a_2\sum_{m_k=1}^{\infty}\frac{m_k^2}{K_{m_k}}\cdot\frac{\cos^2(m_k' a_2)}{\left(m_k^2+\left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(m_k^2-\left(\frac{m a_k}{a_2}\right)^2\right)}\right\} \quad (7)$$

$$\sum_{n_k=2}^{\infty}\frac{Q_{n_k m}}{n_k}\cdot t^{(1)}_{n_k} = -\frac{16}{\pi^2}\cdot\frac{a_k}{a_2}\,(-)^{\frac{m-1}{2}}\left\{\frac{\lambda_2 a_k}{a_2}\sinh\lambda_2' a_2\sum_{n_k=2}^{\infty}\frac{\cos^2(n_k' a_2)}{\left(n_k^2+\left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(n_k^2-\left(\frac{m a_k}{a_2}\right)^2\right)}\right. \quad (8)$$

$$\left. +\cosh\lambda_2' a_2\sum_{n_k=2}^{\infty}\frac{n_k\sin(n_k' a_2)\cos(n_k' a_2)}{\left(n_k^2+\left(\frac{\lambda_2 a_k}{a_2}\right)^2\right)\left(n_k^2-\left(\frac{m a_k}{a_2}\right)^2\right)}\right\} \quad (10)$$

$$\sum_{n_k=2}^{\infty}\frac{P_{n_k n}P_{n_k N}}{K_{n_k}} = \frac{16}{\pi^2}\left(\frac{a_k}{a_2}\right)^2(-)^{\frac{n+N}{2}}\sum_{n_k=2}^{\infty}\frac{n_k^2}{K_{n_k}}\cdot\frac{\sin^2(n_k' a_2)}{\left(n_k^2-\left(\frac{n a_k}{a_2}\right)^2\right)\left(n_k^2-\left(\frac{N a_k}{a_2}\right)^2\right)}$$

$$\sum_{m_k=1}^{\infty}\frac{P_{m_k n}Q_{m_k m}}{m_k} = -\frac{16}{\pi^2}\left(\frac{a_k}{a_2}\right)^2(-)^{\frac{n+m-1}{2}}\sum_{m_k=1}^{\infty}\frac{m_k\cdot\sin(m_k' a_2)\cos(m_k' a_2)}{\left(m_k^2-\left(\frac{n a_k}{a_2}\right)^2\right)\left(m_k^2-\left(\frac{m a_k}{a_2}\right)^2\right)}$$

$$\sum_{n_k=2}^{\infty}\frac{P_{n_k n}Q_{n_k m}}{n_k} = -\frac{16}{\pi^2}\left(\frac{a_k}{a_2}\right)^2(-)^{\frac{n+m-1}{2}}\sum_{n_k=2}^{\infty}\frac{n_k\cdot\sin(n_k' a_2)\cos(n_k' a_2)}{\left(n_k^2-\left(\frac{n a_k}{a_2}\right)^2\right)\left(n_k^2-\left(\frac{m a_k}{a_2}\right)^2\right)}$$

$$\sum_{m_k=1}^{\infty}\frac{Q_{m_k M}Q_{m_k m}}{K_{m_k}} = -\frac{16}{\pi^2}\left(\frac{a_k}{a_2}\right)^2(-)^{\frac{m+M}{2}}\sum_{m_k=1}^{\infty}\frac{m_k^2}{K_{m_k}}\cdot\frac{\cos^2(m_k' a_2)}{\left(m_k^2-\left(\frac{m a_k}{a_2}\right)^2\right)\left(m_k^2-\left(\frac{M a_k}{a_2}\right)^2\right)}$$

Tables 5 and 6 give the numerical value of the part of the infinite series which is contained within the summation sign.

The first set of twelve summations resulting from the interaction of the Kelvin and Poincaré waves are labelled by circled numbers, the order of which will become obvious.

The latter set of four summations resulting from the interaction of the Poincaré waves depends on the values of the integers n, N, m and M. In Table 6 they are therefore to be recognized by the value of $_{j}J$ inserted after the value of a given summation.

Table V

Table V gives the values of the summations arising from the interaction between the Kelvin and the Poincaré waves.

These summations are evaluated sequentially for $j = 0, 1, \ldots 19$ in an order which is convenient for automatic computation.

The even lines give summations ① to ⑥ while the odd lines give ⑦ to ⑫.

There are four sets of such quantities; two for M_2, two for K_1. The first of the pair gives the summations for $k = 1$, while the other gives the summations for $k = 3$.

00	-.61631	.05970	-.35169	-.11938	-.36906	.00577
01	.09544	.00100	-.11646	.08470	-.17994	.03827
02	.02870	.01268	.02491	.02817	.00580	.00384
03	.01446	.00216	.00978	.00994	.01787	.00200
04	.00719	.00297	.00630	.00829	.00146	.00037
05	.00581	.00085	.00329	.00359	.00635	.00065
06	.00319	.00130	.00280	.00402	.00065	.00007
07	.00318	.00044	.00163	.00184	.00323	.00032
08	.00180	.00072	.00158	.00241	.00037	.00001
09	.00203	.00027	.00097	.00111	.00195	.00019
10	.00115	.00046	.00101	.00163	.00024	.00000
11	.00142	.00018	.00064	.00075	.00130	.00012
12	.00080	.00032	.00070	.00120	.00017	.00000
13	.00107	.00013	.00045	.00053	.00093	.00009
14	.00058	.00023	.00052	.00095	.00013	.00000
15	.00086	.00010	.00034	.00042	.00070	.00006
16	.00045	.00017	.00040	.00087	.00012	.00000
17	.00079	.00008	.00025	.00001	.00054	.00005
18	.00035	.00014	.00031	.00058	.00020	.00001
19	.00055	.00006	.00020	.00000	.00043	.00004
00	-.91721	.03979	-.83379	.05402	-.27682	.02957
01	-.12644	.06761	-.47669	.09657	-.51729	.08329
02	-.15329	.05865	-.14607	.05282	.00436	.05147
03	.02174	.01279	.03136	.01134	.03623	.01129
04	.03855	.01108	.03695	.01623	.00110	.00867
05	.00934	.00522	.00983	.00410	.01189	.00467
06	.01715	.00451	.01646	.00809	.00049	.00326
07	.00531	.00275	.00471	.00210	.00587	.00247
08	.00965	.00244	.00926	.00492	.00028	.00168
09	.00347	.00169	.00275	.00127	.00349	.00151

10	.00617	.00153	.00593	.00335	.00018	.00101
11	.00247	.00113	.00180	.00085	.00232	.00102
12	.00429	.00105	.00412	.00244	.00013	.00067
13	.00187	.00001	.00127	.00061	.00165	.00073
14	.00319	.00076	.00302	.00188	.00010	.00048
15	.00147	.00061	.00094	.00046	.00123	.00055
16	.00241	.00058	.00232	.00150	.00008	.00036
17	.00119	.00040	.00073	.00036	.00096	.00043
18	.00190	.00045	.00183	.00123	.00006	.00028
19	.00100	.00038	.00058	.00029	.00076	.00034
00	-.32506	.03769	-.10939	.06599	-.26898	.02897
01	.03863	.00820	.02099	.03113	.04592	.01056
02	.00358	.00165	.00341	.00669	.00106	.00091
03	.00525	.00107	.00190	.00351	.00497	.00103
04	.00089	.00040	.00085	.00201	.00026	.00029
05	.00207	.00039	.00065	.00127	.00178	.00036
06	.00039	.00017	.00038	.00102	.00011	.00014
07	.00114	.00020	.00033	.00067	.00091	.00018
08	.00022	.00009	.00022	.00073	.00001	.00008
09	.00077	.00012	.00019	.00009	.00054	.00011
10	.00014	.00006	.00013	.00039	.00016	.00005
11	.00047	.00007	.00012	.00007	.00036	.00007
12	.00009	.00004	.00009	.00026	.00011	.00004
13	.00033	.00005	.00009	.00005	.00026	.00005
14	.00007	.00003	.00006	.00018	.00008	.00003
15	.00024	.00004	.00006	.00004	.00019	.00004
16	.00005	.00002	.00005	.00014	.00006	.00002
17	.00019	.00003	.00005	.00003	.00015	.00003
18	.00004	.00001	.00004	.00011	.00005	.00001
19	.00015	.00002	.00004	.00002	.00012	.00002

00	-.55315	.05642	-.33037	-.11771	-.42375	.01591
01	.09908	.00515	.09599	.07454	-.13860	.02993
02	.01908	.00844	.01882	.02252	.00168	.00106
03	.01369	.00188	.00831	.00840	.01417	.00204
04	.00477	.00201	.00472	.00654	.00042	.00011
05	.00538	.00071	.00284	.00303	.00506	.00069
06	.00212	.00088	.00210	.00317	.00018	.00010
07	.00291	.00037	.00141	.00155	.00257	.00035
08	.00119	.00049	.00118	.00190	.00010	.00007
09	.00185	.00022	.00084	.00094	.00155	.00021
10	.00076	.00031	.00075	.00129	.00006	.00005
11	.00130	.00015	.00056	.00064	.00104	.00014
12	.00053	.00022	.00052	.00097	.00004	.00004
13	.00099	.00011	.00040	.00048	.00074	.00010
14	.00038	.00016	.00039	.00086	.00002	.00003
15	.00084	.00008	.00029	.00001	.00056	.00007
16	.00029	.00012	.00029	.00056	.00022	.00003
17	.00059	.00006	.00022	.00002	.00043	.00005
18	.00023	.00009	.00023	.00042	.00017	.00002
19	.00045	.00005	.00018	.00002	.00034	.00004

Table VI

Table VI gives the numerical value of the summation arising from the interaction of the Poincaré waves. There are 20^2 such summations if we choose to terminate for $j = 19$.

They are conveniently labelled in terms of j and J which follow these quantities and on which they depend. They are given in the same order as the previous summations. It should be noted that such a type of interaction is almost negligible.

.00772 00 00	-.13719 00 01	.00213 00 02	.01216 00 03
.00044 00 04	.00431 00 05	.00019 00 06	.00219 00 07
.00010 00 08	.00132 00 09	.00006 00 10	.00088 00 11
.00004 00 12	.00063 00 13	.00003 00 14	.00047 00 15
.00002 00 16	.00036 00 17	.00002 00 18	.00029 00 19
.00546 01 00	.04274 01 01	.00161 01 02	.00365 01 03
.00031 01 04	.00130 01 05	.00013 01 06	.00066 01 07
.00007 01 08	.00039 01 09	.00004 01 10	.00026 01 11
.00003 01 12	.00019 01 13	.00002 01 14	.00014 01 15
.00001 01 16	.00011 01 17	.00001 01 18	.00008 01 19
.00213 02 00	.00824 02 01	.00071 02 02	.00060 02 03
.00011 02 04	.00022 02 05	.00005 02 06	.00011 02 07
.00002 02 08	.00006 02 09	.00001 02 10	.00004 02 11
.00001 02 12	.00003 02 13	.00000 02 14	.00002 02 15
.00000 02 16	.00001 02 17	.00000 02 18	.00001 02 19
.00009 03 00	.00365 03 01	.00004 03 02	.00037 03 03
.00001 03 04	.00012 03 05	.00000 03 06	.00006 03 07
.00000 03 08	.00003 03 09	.00000 03 10	.00002 03 11
.00000 03 12	.00001 03 13	.00000 03 14	.00001 03 15
.00000 03 16	.00001 03 17	.00000 03 18	.00000 03 19
.00044 04 00	.00192 04 01	.00011 04 02	.00016 04 03
.00003 04 04	.00005 04 05	.00001 04 06	.00002 04 07
.00000 04 08	.00001 04 09	.00000 04 10	.00001 04 11
.00000 04 12	.00000 04 13	.00000 04 14	.00000 04 15
.00000 04 16	.00000 04 17	.00000 04 18	.00000 04 19
.00002 05 00	.00130 05 01	.00000 05 02	.00012 05 03
.00000 05 04	.00004 05 05	.00000 05 06	.00002 05 07
.00000 05 08	.00001 05 09	.00000 05 10	.00000 05 11
.00000 05 12	.00000 05 13	.00000 05 14	.00000 05 15
.00000 05 16	.00000 05 17	.00000 05 18	.00000 05 19

.00019 06 00	.00004 06 01	.00005 06 02	.00007 06 03
.00001 06 04	.00002 06 05	.00000 06 06	.00001 06 07
.00000 06 08	.00000 06 09	.00000 06 10	.00000 06 11
.00000 06 12	.00000 06 13	.00000 06 14	.00000 06 15
.00000 06 16	.00000 06 17	.00000 06 18	.00000 06 19
.00001 07 00	.00066 07 01	.00000 07 02	.00006 07 03
.00000 07 04	.00002 07 05	.00000 07 06	.00001 07 07
.00000 07 08	.00000 07 09	.00000 07 10	.00000 07 11
.00000 07 12	.00000 07 13	.00000 07 14	.00000 07 15
.00000 07 16	.00000 07 17	.00000 07 18	.00000 07 19
.00010 08 00	.00047 08 01	.00002 08 02	.00004 08 03
.00000 08 04	.00001 08 05	.00000 08 06	.00000 08 07
.00000 08 08	.00000 08 09	.00000 08 10	.00000 08 11
.00000 08 12	.00000 08 13	.00000 08 14	.00000 08 15
.00000 08 16	.00000 08 17	.00000 08 18	.00000 08 19
.00000 09 00	.00039 09 01	.00000 09 02	.00003 09 03
.00000 09 04	.00001 09 05	.00000 09 06	.00000 09 07
.00000 09 08	.00000 09 09	.00000 09 10	.00000 09 11
.00000 09 12	.00000 09 13	.00000 09 14	.00000 09 15
.00000 09 16	.00000 09 17	.00000 09 18	.00000 09 19
.00006 10 00	.00030 10 01	.00001 10 02	.00002 10 03
.00000 10 04	.00000 10 05	.00000 10 06	.00000 10 07
.00000 10 08	.00000 10 09	.00000 10 10	.00000 10 11
.00000 10 12	.00000 10 13	.00000 10 14	.00000 10 15
.00000 10 16	.00000 10 17	.00000 10 18	.00000 10 19
.00000 11 00	.00026 11 01	.00000 11 02	.00002 11 03
.00000 11 04	.00000 11 05	.00000 11 06	.00000 11 07
.00000 11 08	.00000 11 09	.00000 11 10	.00000 11 11
.00000 11 12	.00000 11 13	.00000 11 14	.00000 11 15
.00000 11 16	.00000 11 17	.00000 11 18	.00000 11 19

.00004 12 00	.00021 12 01	.00001 12 02	.00001 12 03
.00000 12 04	.00000 12 05	.00000 12 06	.00000 12 07
.00000 12 08	.00000 12 09	.00000 12 10	.00000 12 11
.00000 12 12	.00000 12 13	.00000 12 14	.00000 12 15
.00000 12 16	.00000 12 17	.00000 12 18	.00000 12 19
.00000 13 00	.00019 13 01	.00000 13 02	.00001 13 03
.00000 13 04	.00000 13 05	.00000 13 06	.00000 13 07
.00000 13 08	.00000 13 09	.00000 13 10	.00000 13 11
.00000 13 12	.00000 13 13	.00000 13 14	.00000 13 15
.00000 13 16	.00000 13 17	.00000 13 18	.00000 13 19
.00003 14 00	.00015 14 01	.00000 14 02	.00001 14 03
.00000 14 04	.00000 14 05	.00000 14 06	.00000 14 07
.00000 14 08	.00000 14 09	.00000 14 10	.00000 14 11
.00000 14 12	.00000 14 13	.00000 14 14	.00000 14 15
.00000 14 16	.00000 14 17	.00000 14 18	.00000 14 19
.00000 15 00	.00014 15 01	.00000 15 02	.00001 15 03
.00000 15 04	.00000 15 05	.00000 15 06	.00000 15 07
.00000 15 08	.00000 15 09	.00000 15 10	.00000 15 11
.00000 15 12	.00000 15 13	.00000 15 14	.00000 15 15
.00000 15 16	.00000 15 17	.00000 15 18	.00000 15 19
.00002 16 00	.00011 16 01	.00000 16 02	.00001 16 03
.00000 16 04	.00000 16 05	.00000 16 06	.00000 16 07
.00000 16 08	.00000 16 09	.00000 16 10	.00000 16 11
.00000 16 12	.00000 16 13	.00000 16 14	.00000 16 15
.00000 16 16	.00000 16 17	.00000 16 18	.00000 16 19
.00000 17 00	.00011 17 01	.00000 17 02	.00001 17 03
.00000 17 04	.00000 17 05	.00000 17 06	.00000 17 07
.00000 17 08	.00000 17 09	.00000 17 10	.00000 17 11
.00000 17 12	.00000 17 13	.00000 17 14	.00000 17 15
.00000 17 16	.00000 17 17	.00000 17 18	.00000 17 19

.00002 18 00	.00009 18 01	.00000 18 02	.00000 18 03
.00000 18 04	.00000 18 05	.00000 18 06	.00000 18 07
.00000 18 08	.00000 18 09	.00000 18 10	.00000 18 11
.00000 18 12	.00000 18 13	.00000 18 14	.00000 18 15
.00000 18 16	.00000 18 17	.00000 18 18	.00000 18 19
.00000 19 00	.00008 19 01	.00000 19 02	.00000 19 03
.00000 19 04	.00000 19 05	.00000 19 06	.00000 19 07
.00000 19 08	.00000 19 09	.00000 19 10	.00000 19 11
.00000 19 12	.00000 19 13	.00000 19 14	.00000 19 15
.00000 19 16	.00000 19 17	.00000 19 18	.00000 19 19
.00260 00 00	-.53094 00 01	.00465 00 02	.02266 00 03
.00058 00 04	.00770 00 05	.00022 00 06	.00387 00 07
.00012 00 08	.00232 00 09	.00007 00 10	.00155 00 11
.00005 00 12	.00111 00 13	.00003 00 14	.00083 00 15
.00002 00 16	.00064 00 17	.00002 00 18	.00051 00 19
.00597 01 00	-.31042 01 01	.01103 01 02	.01237 01 03
.00119 01 04	.00432 01 05	.00047 01 06	.00218 01 07
.00025 01 08	.00131 01 09	.00016 01 10	.00088 01 11
.00011 01 12	.00062 01 13	.00008 01 14	.00047 01 15
.00006 01 16	.00036 01 17	.00004 01 18	.00029 01 19
.00465 02 00	-.10396 02 01	.00896 02 02	.00302 02 03
.00077 02 04	.00126 02 05	.00032 02 06	.00065 02 07
.00018 02 08	.00039 02 09	.00011 02 10	.00026 02 11
.00007 02 12	.00018 02 13	.00005 02 14	.00014 02 15
.00004 02 16	.00011 02 17	.00003 02 18	.00008 02 19
.00134 03 00	.01237 03 01	.00300 03 02	.00159 03 03
.00001 03 04	.00030 03 05	.00005 03 06	.00015 03 07
.00003 03 08	.00008 03 09	.00002 03 10	.00005 03 11
.00001 03 12	.00004 03 13	.00001 03 14	.00003 03 15
.00000 03 16	.00002 03 17	.00000 03 18	.00001 03 19

.00058 04 00	.02239 04 01	.00077 04 02	.00126 04 03
.00031 04 04	.00028 04 05	.00006 04 06	.00016 04 07
.00003 04 08	.00009 04 09	.00002 04 10	.00006 04 11
.00001 04 12	.00004 04 13	.00001 04 14	.00003 04 15
.00000 04 16	.00002 04 17	.00000 04 18	.00002 04 19
.00042 05 00	.00432 05 01	.00078 05 02	.00030 05 03
.00014 05 04	.00015 05 05	.00001 05 06	.00004 05 07
.00001 05 08	.00002 05 09	.00000 05 10	.00001 05 11
.00000 05 12	.00001 05 13	.00000 05 14	.00001 05 15
.00000 05 16	.00000 05 17	.00000 05 18	.00000 05 19
.00022 06 00	.00972 06 01	.00032 06 02	.00049 06 03
.00006 06 04	.00017 06 05	.00004 06 06	.00006 06 07
.00001 06 08	.00004 06 09	.00000 06 10	.00002 06 11
.00000 06 12	.00002 06 13	.00000 06 14	.00001 06 15
.00000 06 16	.00001 06 17	.00000 06 18	.00000 06 19
.00020 07 00	.00218 07 01	.00037 07 02	.00015 07 03
.00005 07 04	.00004 07 05	.00002 07 06	.00003 07 07
.00000 07 08	.00001 07 09	.00000 07 10	.00000 07 11
.00000 07 12	.00000 07 13	.00000 07 14	.00000 07 15
.00000 07 16	.00000 07 17	.00000 07 18	.00000 07 19
.00012 08 00	.00542 08 01	.00018 08 02	.00027 08 03
.00003 08 04	.00008 08 05	.00001 08 06	.00004 08 07
.00001 08 08	.00002 08 09	.00000 08 10	.00001 08 11
.00000 08 12	.00001 08 13	.00000 08 14	.00000 08 15
.00000 08 16	.00000 08 17	.00000 08 18	.00000 08 19
.00012 09 00	.00131 09 01	.00022 09 02	.00008 09 03
.00003 09 04	.00002 09 05	.00001 09 06	.00001 09 07
.00000 09 08	.00001 09 09	.00000 09 10	.00000 09 11
.00000 09 12	.00000 09 13	.00000 09 14	.00000 09 15
.00000 09 16	.00000 09 17	.00000 09 18	.00000 09 19

.00007 10 00	.00346 10 01	.00011 10 02	.00017 10 03
.00002 10 04	.00005 10 05	.00000 10 06	.00002 10 07
.00000 10 00	.00001 10 09	.00000 10 10	.00000 10 11
.00000 10 12	.00000 10 13	.00000 10 14	.00000 10 15
.00000 10 16	.00000 10 17	.00000 10 18	.00000 10 19
.00008 11 00	.00008 11 01	.00014 11 02	.00005 11 03
.00002 11 04	.00001 11 05	.00000 11 06	.00000 11 07
.00000 11 08	.00000 11 09	.00000 11 10	.00000 11 11
.00000 11 12	.00000 11 13	.00000 11 14	.00000 11 15
.00000 11 16	.00000 11 17	.00000 11 18	.00000 11 19
.00005 12 00	.00239 12 01	.00007 12 02	.00011 12 03
.00001 12 04	.00003 12 05	.00000 12 06	.00001 12 07
.00000 12 08	.00001 12 09	.00000 12 10	.00000 12 11
.00000 12 12	.00000 12 13	.00000 12 14	.00000 12 15
.00000 12 16	.00000 12 17	.00000 12 18	.00000 12 19
.00005 13 00	.00062 13 01	.00010 13 02	.00004 13 03
.00001 13 04	.00001 13 05	.00000 13 06	.00000 13 07
.00000 13 08	.00000 13 09	.00000 13 10	.00000 13 11
.00000 13 12	.00000 13 13	.00000 13 14	.00000 13 15
.00000 13 16	.00000 13 17	.00000 13 18	.00000 13 19
.00003 14 00	.00176 14 01	.00005 14 02	.00008 14 03
.00001 14 04	.00002 14 05	.00000 14 06	.00001 14 07
.00000 14 08	.00000 14 09	.00000 14 10	.00000 14 11
.00000 14 12	.00000 14 13	.00000 14 14	.00000 14 15
.00000 14 16	.00000 14 17	.00000 14 18	.00000 14 19
.00004 15 00	.00047 15 01	.00007 15 02	.00003 15 03
.00001 15 04	.00001 15 05	.00000 15 06	.00000 15 07
.00000 15 08	.00000 15 09	.00000 15 10	.00000 15 11
.00000 15 12	.00000 15 13	.00000 15 14	.00000 15 15
.00000 15 16	.00000 15 17	.00000 15 18	.00000 15 19

.00002 16 00	.00134 16 01	.00004 16 02	.00006 16 03
.00000 16 04	.00002 16 05	.00000 16 06	.00001 16 07
.00000 16 08	.00000 16 09	.00000 16 10	.00000 16 11
.00000 16 12	.00000 16 13	.00000 16 14	.00000 16 15
.00000 16 16	.00000 16 17	.00000 16 18	.00000 16 19
.00003 17 00	.00036 17 01	.00006 17 02	.00002 17 03
.00000 17 04	.00000 17 05	.00000 17 06	.00000 17 07
.00000 17 08	.00000 17 09	.00000 17 10	.00000 17 11
.00000 17 12	.00000 17 13	.00000 17 14	.00000 17 15
.00000 17 16	.00000 17 17	.00000 17 18	.00000 17 19
.00002 18 00	.00106 18 01	.00003 18 02	.00005 18 03
.00000 18 04	.00001 18 05	.00000 18 06	.00000 18 07
.00000 18 08	.00000 18 09	.00000 18 10	.00000 18 11
.00000 18 12	.00000 18 13	.00000 18 14	.00000 18 15
.00000 18 16	.00000 18 17	.00000 18 18	.00000 18 19
.00002 19 00	.00029 19 01	.00004 19 02	.00001 19 03
.00000 19 04	.00000 19 05	.00000 19 06	.00000 19 07
.00000 19 08	.00000 19 09	.00000 19 10	.00000 19 11
.00000 19 12	.00000 19 13	.00000 19 14	.00000 19 15
.00000 19 16	.00000 19 17	.00000 19 18	.00000 19 19
.00353 00 00	.01919 00 01	.00021 00 02	.00202 00 03
.00005 00 04	.00072 00 05	.00002 00 06	.00036 00 07
.00001 00 08	.00022 00 09	.00000 00 10	.00014 00 11
.00000 00 12	.00010 00 13	.00000 00 14	.00008 00 15
.00000 00 16	.00006 00 17	.00000 00 18	.00005 00 19
.00111 01 00	.00293 01 01	.00007 01 02	.00030 01 03
.00001 01 04	.00010 01 05	.00000 01 06	.00005 01 07
.00000 01 08	.00003 01 09	.00000 01 10	.00002 01 11
.00000 01 12	.00001 01 13	.00000 01 14	.00001 01 15
.00000 01 16	.00000 01 17	.00000 01 18	.00000 01 19

.00021 02 00	.00029 02 01	.00001 02 02	.00002 02 03
.00000 02 04	.00001 02 05	.00000 02 06	.00000 02 07
.00000 02 08	.00000 02 09	.00000 02 10	.00000 02 11
.00000 02 12	.00000 02 13	.00000 02 14	.00000 02 15
.00000 02 16	.00000 02 17	.00000 02 18	.00000 02 19
.00009 03 00	.00030 03 01	.00000 03 02	.00003 03 03
.00000 03 04	.00001 03 05	.00000 03 06	.00000 03 07
.00000 03 08	.00000 03 09	.00000 03 10	.00000 03 11
.00000 03 12	.00000 03 13	.00000 03 14	.00000 03 15
.00000 03 16	.00000 03 17	.00000 03 18	.00000 03 19
.00005 04 00	.00007 04 01	.00000 04 02	.00000 04 03
.00000 04 04	.00000 04 05	.00000 04 06	.00000 04 07
.00000 04 08	.00000 04 09	.00000 04 10	.00000 04 11
.00000 04 12	.00000 04 13	.00000 04 14	.00000 04 15
.00000 04 16	.00000 04 17	.00000 04 18	.00000 04 19
.00003 05 00	.00010 05 01	.00000 05 02	.00001 05 03
.00000 05 04	.00000 05 05	.00000 05 06	.00000 05 07
.00000 05 08	.00000 05 09	.00000 05 10	.00000 05 11
.00000 05 12	.00000 05 13	.00000 05 14	.00000 05 15
.00000 05 16	.00000 05 17	.00000 05 18	.00000 05 19
.00002 06 00	.00003 06 01	.00000 06 02	.00000 06 03
.00000 06 04	.00000 06 05	.00000 06 06	.00000 06 07
.00000 06 08	.00000 06 09	.00000 06 10	.00000 06 11
.00000 06 12	.00000 06 13	.00000 06 14	.00000 06 15
.00000 06 16	.00000 06 17	.00000 06 18	.00000 06 19
.00001 07 00	.00005 07 01	.00000 07 02	.00000 07 03
.00000 07 04	.00000 07 05	.00000 07 06	.00000 07 07
.00000 07 08	.00000 07 09	.00000 07 10	.00000 07 11
.00000 07 12	.00000 07 13	.00000 07 14	.00000 07 15
.00000 07 16	.00000 07 17	.00000 07 18	.00000 07 19

.00001 08 00	.00001 08 01	.00000 08 02	.00000 08 03
.00000 08 04	.00000 08 05	.00000 08 06	.00000 08 07
.00000 08 08	.00000 08 09	.00000 08 10	.00000 08 11
.00000 08 12	.00000 08 13	.00000 08 14	.00000 08 15
.00000 08 16	.00000 08 17	.00000 08 18	.00000 08 19
.00001 09 00	.00003 09 01	.00000 09 02	.00000 09 03
.00000 09 04	.00000 09 05	.00000 09 06	.00000 09 07
.00000 09 08	.00000 09 09	.00000 09 10	.00000 09 11
.00000 09 12	.00000 09 13	.00000 09 14	.00000 09 15
.00000 09 16	.00000 09 17	.00000 09 18	.00000 09 19
.00000 10 00	.00001 10 01	.00000 10 02	.00000 10 03
.00000 10 04	.00000 10 05	.00000 10 06	.00000 10 07
.00000 10 08	.00000 10 09	.00000 10 10	.00000 10 11
.00000 10 12	.00000 10 13	.00000 10 14	.00000 10 15
.00000 10 16	.00000 10 17	.00000 10 18	.00000 10 19
.00000 11 00	.00002 11 01	.00000 11 02	.00000 11 03
.00000 11 04	.00000 11 05	.00000 11 06	.00000 11 07
.00000 11 08	.00000 11 09	.00000 11 10	.00000 11 11
.00000 11 12	.00000 11 13	.00000 11 14	.00000 11 15
.00000 11 16	.00000 11 17	.00000 11 18	.00000 11 19
.00000 12 00	.00000 12 01	.00000 12 02	.00000 12 03
.00000 12 04	.00000 12 05	.00000 12 06	.00000 12 07
.00000 12 08	.00000 12 09	.00000 12 10	.00000 12 11
.00000 12 12	.00000 12 13	.00000 12 14	.00000 12 15
.00000 12 16	.00000 12 17	.00000 12 18	.00000 12 19
.00000 13 00	.00001 13 01	.00000 13 02	.00000 13 03
.00000 13 04	.00000 13 05	.00000 13 06	.00000 13 07
.00000 13 08	.00000 13 09	.00000 13 10	.00000 13 11
.00000 13 12	.00000 13 13	.00000 13 14	.00000 13 15
.00000 13 16	.00000 13 17	.00000 13 18	.00000 13 19

.00000 14 00	.00000 14 01	.00000 14 02	.00000 14 03
.00000 14 04	.00000 14 05	.00000 14 06	.00000 14 07
.00000 14 08	.00000 14 09	.00000 14 10	.00000 14 11
.00000 14 12	.00000 14 13	.00000 14 14	.00000 14 15
.00000 14 16	.00000 14 17	.00000 14 18	.00000 14 19
.00000 15 00	.00001 15 01	.00000 15 02	.00000 15 03
.00000 15 04	.00000 15 05	.00000 15 06	.00000 15 07
.00000 15 08	.00000 15 09	.00000 15 10	.00000 15 11
.00000 15 12	.00000 15 13	.00000 15 14	.00000 15 15
.00000 15 16	.00000 15 17	.00000 15 18	.00000 15 19
.00000 16 00	.00000 16 01	.00000 16 02	.00000 16 03
.00000 16 04	.00000 16 05	.00000 16 06	.00000 16 07
.00000 16 08	.00000 16 09	.00000 16 10	.00000 16 11
.00000 16 12	.00000 16 13	.00000 16 14	.00000 16 15
.00000 16 16	.00000 16 17	.00000 16 18	.00000 16 19
.00000 17 00	.00000 17 01	.00000 17 02	.00000 17 03
.00000 17 04	.00000 17 05	.00000 17 06	.00000 17 07
.00000 17 08	.00000 17 09	.00000 17 10	.00000 17 11
.00000 17 12	.00000 17 13	.00000 17 14	.00000 17 15
.00000 17 16	.00000 17 17	.00000 17 18	.00000 17 19
.00000 18 00	.00000 18 01	.00000 18 02	.00000 18 03
.00000 18 04	.00000 18 05	.00000 18 06	.00000 18 07
.00000 18 08	.00000 18 09	.00000 18 10	.00000 18 11
.00000 18 12	.00000 18 13	.00000 18 14	.00000 18 15
.00000 18 16	.00000 18 17	.00000 18 18	.00000 18 19
.00000 19 00	.00000 19 01	.00000 19 02	.00000 19 03
.00000 19 04	.00000 19 05	.00000 19 06	.00000 19 07
.00000 19 08	.00000 19 09	.00000 19 10	.00000 19 11
.00000 19 12	.00000 19 13	.00000 19 14	.00000 19 15
.00000 19 16	.00000 19 17	.00000 19 18	.00000 19 19

.00713 00 00	.09304 00 01	.00141 00 02	.00875 00 03
.00030 00 04	.00311 00 05	.00013 00 06	.00158 00 07
.00007 00 08	.00095 00 09	.00004 00 10	.00064 00 11
.00003 00 12	.00045 00 13	.00002 00 14	.00034 00 15
.00001 00 16	.00026 00 17	.00001 00 18	.00021 00 19
.00419 01 00	.02498 01 01	.00089 01 02	.00227 01 03
.00018 01 04	.00081 01 05	.00007 01 06	.00041 01 07
.00004 01 08	.00024 01 09	.00002 01 10	.00016 01 11
.00001 01 12	.00011 01 13	.00001 01 14	.00008 01 15
.00001 01 16	.00006 01 17	.00000 01 18	.00005 01 19
.00141 02 00	.00420 02 01	.00035 02 02	.00033 02 03
.00006 02 04	.00012 02 05	.00002 02 06	.00006 02 07
.00001 02 08	.00003 02 09	.00000 02 10	.00002 02 11
.00000 02 12	.00001 02 13	.00000 02 14	.00001 02 15
.00000 02 16	.00001 02 17	.00000 02 18	.00000 02 19
.00016 03 00	.00227 03 01	.00000 03 02	.00023 03 03
.00000 03 04	.00008 03 05	.00000 03 06	.00004 03 07
.00000 03 08	.00002 03 09	.00000 03 10	.00001 03 11
.00000 03 12	.00001 03 13	.00000 03 14	.00000 03 15
.00000 03 16	.00000 03 17	.00000 03 18	.00000 03 19
.00030 04 00	.00099 04 01	.00006 04 02	.00009 04 03
.00001 04 04	.00003 04 05	.00000 04 06	.00001 04 07
.00000 04 08	.00000 04 09	.00000 04 10	.00000 04 11
.00000 04 12	.00000 04 13	.00000 04 14	.00000 04 15
.00000 04 16	.00000 04 17	.00000 04 18	.00000 04 19
.00005 05 00	.00081 05 01	.00000 05 02	.00008 05 03
.00000 05 04	.00002 05 05	.00000 05 06	.00001 05 07
.00000 05 08	.00000 05 09	.00000 05 10	.00000 05 11
.00000 05 12	.00000 05 13	.00000 05 14	.00000 05 15
.00000 05 16	.00000 05 17	.00000 05 18	.00000 05 19

.00013 06 00	.00043 06 01	.00002 06 02	.00003 06 03
.00000 06 04	.00001 06 05	.00000 06 06	.00000 06 07
.00000 06 08	.00000 06 09	.00000 06 10	.00000 06 11
.00000 06 12	.00000 06 13	.00000 06 14	.00000 06 15
.00000 06 16	.00000 06 17	.00000 06 18	.00000 06 19
.00002 07 00	.00041 07 01	.00000 07 02	.00004 07 03
.00000 07 04	.00001 07 05	.00000 07 06	.00000 07 07
.00000 07 08	.00000 07 09	.00000 07 10	.00000 07 11
.00000 07 12	.00000 07 13	.00000 07 14	.00000 07 15
.00000 07 16	.00000 07 17	.00000 07 18	.00000 07 19
.00007 08 00	.00024 08 01	.00001 08 02	.00002 08 03
.00000 08 04	.00000 08 05	.00000 08 06	.00000 08 07
.00000 08 08	.00000 08 09	.00000 08 10	.00000 08 11
.00000 08 12	.00000 08 13	.00000 08 14	.00000 08 15
.00000 08 16	.00000 08 17	.00000 08 18	.00000 08 19
.00001 09 00	.00024 09 01	.00000 09 02	.00002 09 03
.00000 09 04	.00000 09 05	.00000 09 06	.00000 09 07
.00000 09 08	.00000 09 09	.00000 09 10	.00000 09 11
.00000 09 12	.00000 09 13	.00000 09 14	.00000 09 15
.00000 09 16	.00000 09 17	.00000 09 18	.00000 09 19
.00004 10 00	.00015 10 01	.00000 10 02	.00001 10 03
.00000 10 04	.00000 10 05	.00000 10 06	.00000 10 07
.00000 10 08	.00000 10 09	.00000 10 10	.00000 10 11
.00000 10 12	.00000 10 13	.00000 10 14	.00000 10 15
.00000 10 16	.00000 10 17	.00000 10 18	.00000 10 19
.00001 11 00	.00016 11 01	.00000 11 02	.00001 11 03
.00000 11 04	.00000 11 05	.00000 11 06	.00000 11 07
.00000 11 08	.00000 11 09	.00000 11 10	.00000 11 11
.00000 11 12	.00000 11 13	.00000 11 14	.00000 11 15
.00000 11 16	.00000 11 17	.00000 11 18	.00000 11 19

.00003 12 00	.00010 12 01	.00000 12 02	.00000 12 03
.00000 12 04	.00000 12 05	.00000 12 06	.00000 12 07
.00000 12 08	.00000 12 09	.00000 12 10	.00000 12 11
.00000 12 12	.00000 12 13	.00000 12 14	.00000 12 15
.00000 12 16	.00000 12 17	.00000 12 18	.00000 12 19
.00000 13 00	.00011 13 01	.00000 13 02	.00001 13 03
.00000 13 04	.00000 13 05	.00000 13 06	.00000 13 07
.00000 13 08	.00000 13 09	.00000 13 10	.00000 13 11
.00000 13 12	.00000 13 13	.00000 13 14	.00000 13 15
.00000 13 16	.00000 13 17	.00000 13 18	.00000 13 19
.00002 14 00	.00008 14 01	.00000 14 02	.00000 14 03
.00000 14 04	.00000 14 05	.00000 14 06	.00000 14 07
.00000 14 08	.00000 14 09	.00000 14 10	.00000 14 11
.00000 14 12	.00000 14 13	.00000 14 14	.00000 14 15
.00000 14 16	.00000 14 17	.00000 14 18	.00000 14 19
.00000 15 00	.00008 15 01	.00000 15 02	.00000 15 03
.00000 15 04	.00000 15 05	.00000 15 06	.00000 15 07
.00000 15 08	.00000 15 09	.00000 15 10	.00000 15 11
.00000 15 12	.00000 15 13	.00000 15 14	.00000 15 15
.00000 15 16	.00000 15 17	.00000 15 18	.00000 15 19
.00001 16 00	.00006 16 01	.00000 16 02	.00000 16 03
.00000 16 04	.00000 16 05	.00000 16 06	.00000 16 07
.00000 16 08	.00000 16 09	.00000 16 10	.00000 16 11
.00000 16 12	.00000 16 13	.00000 16 14	.00000 16 15
.00000 16 16	.00000 16 17	.00000 16 18	.00000 16 19
.00000 17 00	.00006 17 01	.00000 17 02	.00000 17 03
.00000 17 04	.00000 17 05	.00000 17 06	.00000 17 07
.00000 17 08	.00000 17 09	.00000 17 10	.00000 17 11
.00000 17 12	.00000 17 13	.00000 17 14	.00000 17 15
.00000 17 16	.00000 17 17	.00000 17 18	.00000 17 19

.00001 18 00	.00004 18 01	.00000 18 02	.00000 18 03
.00000 18 04	.00000 18 05	.00000 18 06	.00000 18 07
.00000 18 08	.00000 18 09	.00000 18 10	.00000 18 11
.00000 18 12	.00000 18 13	.00000 18 14	.00000 18 15
.00000 18 16	.00000 18 17	.00000 18 18	.00000 18 19
.00000 19 00	.00005 19 01	.00000 19 02	.00000 19 03
.00000 19 04	.00000 19 05	.00000 19 06	.00000 19 07
.00000 19 08	.00000 19 09	.00000 19 10	.00000 19 11
.00000 19 12	.00000 19 13	.00000 19 14	.00000 19 15
.00000 19 16	.00000 19 17	.00000 19 18	.00000 19 19

The Boundary Condition on v at the Mouth

We have not yet specified the type of motion we wish to assume at the mouth. There is no observational material from which we can draw and we will therefore use the simplest boundary condition we can think of:

$$\begin{aligned} \theta_o &= -i \\ B_n^{(1)} &= D_m^{(1)} = 0 \end{aligned} \tag{59}$$

i.e. we assume the velocity at the mouth to be of a purely Kelvin type with the maximum inward current running approximately $\frac{1}{4}$ period before high water.

Once the calculations will be concluded we will find out that this boundary condition leads to a cotidal chart for M_2 which is in complete qualitative agreement with the chart shown in Fig. 2 based exclusively on coastal observations.

Our assumption therefore must have some physical justification.

Table VII

Numerical Values of the Matrices and of the Column Vectors

All the matrices and column vectors have been multiplied by the factor 10^4; the exceptions are the inverse matrices which have their natural values.

The values for M_2 at the lower and upper boundaries are given first, then those for K_1.

All the matrices and the column vectors have been renormalized so that the diagonal elements of the Gamma matrices have absolute value 1.

```
      -   12       8      -      6      5     4
 2  4501- 10000-  3741   60    749-   21-   320    11   178-    7
 -   113     5     78-    3-    57     2     44-    2-   35     2
 3  2179-  7626- 10000- 4917     4  1173-     1-  548     1   320
 -     1-  211           150     -   112           87     -    68
 4  1537-   193   7312  10000  5218-    7-  1354     4   666-    2
 -   402      2    271     2-   195          148    -   117   749
 5  1372-  3124-     7   6953  10000  5418      -  1480
       -    465           319     -   232          178     -   141
 6   898-   115   2906     12-  6781- 10000- 5549     3  1566
 -   814           514      -   356          263    -   204
 7   784-  2010-     4   2705     -  6659- 10000- 5637        1628
       -    862           550     -   386          290     -   227
 8   633-    82   1899      9-  2592         6578 10000  5700
 -  1677           902      -   581          412         312
 9   584-  1491          1773     -  2506          6513 10000  5745
       -   1714           931     -   604           436     -   327
10   479-    61   1427      6-  1689         2441     -  6466- 10000
 -  5786          1745      -   953          630     -   452
11   477-  1183-     6   1337     -  1628          2396     -  6427
 - 10000-  5814          1771     -   978           644     -   471
12   395-    54   1151      7-  1287         1580     -  2357
    6396  10000   5838      -  1792          995     -   661
13   394-   981          1085     -  1234          1539     -  2327
           6372  10000   5866     -  1807          1011     -   677
14   322-    40    966      8-  1039         1192     -  1514
    2303      -   6353- 10000-  5878         1828     -  1023
15   364-   841           911     -   997          1162     -  1483
           2281      -   6331- 10000- 5898          1839     -  1041
16   269-    37    826      -   873          966     -  1133
    1467      -   2256          6314  10000  5905     -  1848
17   327-   733           793     -   842           942     -  1120
           1447      -   2250         6303  10000  5927     -  1863
18   284-    32    726      -   758          821     -   926
    1095      -   1432          2232     -  6295- 10000- 5926
19   312-   658           702     -   736           803     -   903
           1081      -   1416         2218     -  6276- 10000- 5942
20   247-    35    647      -   671          718     -   776
     894      -   1071          1400     -  2212         6271 10000
```

					GAMMA+	M2					
1		10000-	753	13	135-	2-	49	1	25	-	15
			10	-	7		6	-	4		3
2	-	932-	10000	2943	107-	590-	37	253	19-	140-	11
		89	8-	62-	5	46	4-	35-	3	27	3
3		1251	5650-	10000	4856	17-	1169-	7	546	4-	319
	-	2	211	2-	149-	1	110	1-	86		69
4	-	61-	425-	7186	10000-	5211-		5	6	665	
		401	3-	270-	2	194	1-	148	1	116	
5		646	2229-	35-	6911	10000-	5432-	2	1487	1-	754
			467	-	322		235	-	178		141
6		28-	251-	2861	19	6768-	10000	5552	3-	1574	2
		820	1-	517		356	-	263		204	
7		431	1423-	22-	2678	4	6649-	10000	5639	-	1631
			866	-	550		386	-	290		227
8		48	178-	1867	13	2561-	4-	6573	10000-	5693	
		1677	-	901		581	-	412		312	
9		322	1055-	15-	1753	5	2496	-	6508	10000-	5745
			1714	-	931		604	-	436		327
10		45-	139-	1405	11	1683-	6-	2441		6466-	10000
		5786	-	1745		959	-	630		452	
11		254	836-	12-	1325		1622	-	2396		6427
	-	10000	5814	-	1771		978	-	644		471
12		48-	109-	1131	7	1281	-	1580		2357	
	-	6396	10000-	5845		1792	-	995		661	
13		203	698-	7-	1070		1226	-	1545		2325
		-	6367	10000-	5862		1805	-	1010		676
14		48-	97-	950	8	1039	-	1192		1514	
	-	2303		6353-	10000	5878	-	1828		1023	
15		182	590-	9-	893		997	-	1162		1483
		-	2272		6331-	10000	5898	-	1839		1041
16		46-	84-	743	9	873	-	966		1142	
	-	1467		2256	-	6314	10000-	5905		1848	
17		159	515-	10-	783		842	-	942		1120
		-	1447		2250	-	6303	10000-	5927		1863
18		53-	74-	716		758	-	821		926	
	-	1095		1432	-	2232		6284-	10000	5926	
19		145	457-	11-	691		736	-	803		903
		-	1081		1416	-	2218		6276-	10000	5942
20		47-	59-	635		671	-	694		776	
	-	894		1071	-	1412		2212	-	6271	10000

	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.0040	-.0114	.0122	.0085
	-.0062	-.0048	.0001	-.0039	.0023	-.0026	.0039	.0004	.0028	.0039
2	.4472	-2.0743	1.6087	1.0062	-.1849	.2108	-.1162	.0364	.0340	.1723
	-.1303	-.0020	-.0272	.0226	-.0013	.0689	-.0198	.1644	-.2839	-.2488
3	-.0467	3.4333	-5.3387	-4.0935	1.8701	.1295	.3603	.4650	-.9345	-1.3755
	.9469	.2358	.0752	.0209	-.1222	-.2810	-.0883	-.9310	1.4032	1.1279
4	-.1319	-3.3032	6.1458	7.5926	-5.5187	-2.9007	.3862	-1.8970	3.6171	4.0077
	-2.3288	-.6413	-.0171	-.0787	.4421	.6338	.4123	2.2487	-3.1070	-2.3111
5	-.0306	.8117	-3.1781	-6.3190	9.0398	7.8858	-3.6014	2.5607	-6.8911	-7.1876
	3.6066	.6419	.2092	.0678	-.3945	-.6339	-.4557	-2.3354	3.1459	2.1992
6	.2471	1.4030	-1.2803	1.8682	-7.3423	-11.2828	.1450	.6202	5.9593	7.2183
	-3.3474	.1805	-.8748	.1009	-.4823	.0717	-.4104	-.0175	-.4213	-.4667
7	-.2050	-1.1028	2.5053	1.9619	2.1784	7.8335	-10.82	-7.0028	.5583	-3.6900
	2.8222	-.4518	1.9214	.3233	1.6947	1.1795	1.6514	3.9924	-4.0706	-1.9978
8	-.0765	-.6555	.1387	-1.4681	2.1097	-.9874	7.1472	11.79	-9.0769	-2.6714
	-1.7052	-.7491	-1.7762	-1.1374	-1.7535	-2.0085	-1.9837	-6.2244	6.8310	3.3125
9	.2227	1.1646	-2.2517	-1.2538	-1.9236	-3.2221	-1.1236	-8.6574	12.72	8.3885
	-1.5265	1.5833	-.3426	.8451	.2382	1.0282	.8843	3.7361	-4.5138	-2.1459
10	-.0694	.0308	1.0032	1.8493	-.7255	1.8378	-2.3665	1.5935	-7.5268	-10.44
	6.6784	1.1644	1.7278	1.0357	.5430	.8012	-.1181	.2132	.0323	.0435
11	-.1274	-.9059	1.3093	.2448	1.5236	1.3899	1.2704	2.8849	.3489	6.7020
		-7.0019	.9948	-1.9853	.7304	-1.0456	.4252	-1.4337	2.1142	1.0308
12	.0877	.2548	-1.2261	-1.5700	.1773	-1.6154	1.0932	-1.6385	2.5105	-1.6177
	8.0339	11.14	-7.0158	-1.1735	-1.7405	-.9928	-.4247	-.4492	-.2882	-.3095
13	.0923	.6764	-.6686	.3306	-1.3183	-.6624	-1.1147	-1.4114	-.5440	-1.4792
	-1.8725	-7.9091	11.23	7.2837	-1.1105	2.1051	-.9456	1.3869	-1.9475	-.8680
14	-.1298	-.4555	1.2592	1.2035	.3035	1.5508	-.5498	1.5767	-1.4999	1.2685
	-2.5220	1.4305	-7.9910	-11.22	7.0549	1.1815	1.6128	.7043	.8186	.7228
15	-.0131	-.3783	.0896	-.7183	.9813	.0075	1.0475	.5798	.7202	.3329
	1.8711	2.4930	1.4992	7.7781	-11.10	-7.3245	1.3688	-2.2340	2.1624	.6902
16	.1198	.4808	-1.0473	-.7152	-.6221	-1.3072	.1161	-1.4401	.9631	-.9000
	.9980	-1.4709	2.4957	-1.4586	7.8901	11.03	-6.8911	-1.0286	-2.0124	-1.3706
17	-.0623	.1574	.2931	.8331	-.5323	.5338	-.9040	.1010	-.9140	.1244
	-1.7091	-1.0720	-1.6013	-2.3106	-1.6721	-7.5674	10.52	7.3045	-2.7399	-.1487
18	-.0441	-.4966	.8153	.3108	.7239	.8776	.2692	1.1431	-.4315	.6043
	-.0119	1.4857	-1.0087	1.6031	-2.3810	1.6639	-7.8355	-10.98	7.6522	2.7550
19	.0673	.2011	-.8127	-.9473	-.0110	-1.0346	.5868	-.8326	1.0230	-.5024
	1.3137	-.1754	1.7402	.4581	2.2518	1.8189	3.0042	7.7903	-8.4141	-3.9138
20	-.0214	.1043	.1592	.4994	-.3459	.2802	-.5234	.0273	-.4889	.0614
	-.8548	-.5139	-.6970	-.8840	-.5347	-1.4991	.0380	-2.7726	4.0313	3.1902

GAMMA + M2

1	.9947	-.0859	-.0125	-.0262	-.0395	.0353	.0192	.0079	.0286	-.0262
	-.0095	-.0052	-.0063	-.0007	-.0038	.0003	-.0030	.0017	-.0009	.0013
2	-.0420	-1.4059	-.6991	.4663	.3816	-.4735	-.4277	-.0354	-.5305	.5413
	.1784	.1450	.1361	.0438	.1009	-.0048	.0655	-.0258	.0343	-.0338
3	.1598	-1.3766	-2.6851	2.6894	3.3318	-3.5660	-2.3838	-.6677	-3.1845	2.8886
	.8800	.7243	.6479	.2187	.4806	-.0264	.3007	-.1182	.1564	-.1541
4	.0863	-.8553	-2.4264	6.0229	9.1430	-10.08	-6.0655	-2.0997	-7.7458	6.3320
	1.5452	1.6664	1.1766	.6069	.9219	.0743	.6083	-.1361	.3820	-.3095
5	-.0655	.4296	-.9219	6.0101	13.01	-14.38	-8.4877	-2.3009	-8.8731	6.5983
	.9550	2.0341	.9103	.9384	.8447	.3342	.6423	.0543	.5320	-.3245
6	-.1061	1.2594	1.3180	2.6583	9.4825	-13.76	-9.6760	.9908	-5.0225	4.1335
	.1797	1.7171	.5027	.9296	.6096	.4308	.5470	.1463	.5188	-.2753
7	-.0100	.8082	1.6773	-1.3272	1.8263	-6.3599	-9.1725	6.9785	2.9133	.1777
	.5575	.2657	.4698	.0470	.3347	-.0679	.2103	-.1219	.0890	-.1095
8	.0744	-.0211	.1567	-2.0579	-3.1787	.3888	-5.7317	11.07	9.4624	-3.5835
	1.1467	-1.6374	.2417	-1.0744	-.1856	-.6499	-.3479	-.3526	-.4834	.1655
9	.0413	-.1483	-.9394	.1150	-1.8917	2.0797	-1.9308	7.8282	10.78	-6.8846
	-1.1473	-1.8144	-.9216	-.7994	-.8232	-.2394	-.6100	.0317	-.4399	.3017
10	-.0391	.3281	-.2853	1.4507	1.5549	.0172	.9254	1.3190	6.0346	-9.4213
	-6.6792	1.6415	-1.5405	1.1451	-.5335	.9059	.0224	.5809	.3592	.0182
11	-.0463	.3916	.7303	.2850	1.7070	-1.1909	1.4301	-2.5715	1.1263	-7.5608
		7.2605	1.1987	1.9912	1.0356	.8572	.9583	.1782	.7492	-.3881
12	.0184	-.1134	.4808	-1.1166	-.8259	-.0585	.1342	-1.3288	-1.0648	-2.7178
	-8.1829	10.80	6.9089	-1.3174	1.6888	-1.0520	.5636	-.8695	-.1705	-.2209
13	.0525	-.3724	-.4631	-.5815	-1.6482	.9756	-.9056	1.3454	-.6514	1.8390
	-1.4832	7.5851	11.09	-7.3754	-1.3507	-1.9986	-1.2591	-.6856	-1.1847	.4675
14	.0111	-.0067	-.5312	.7380	.1709	.2758	-.4943	1.3710	.2881	2.3093
	2.7759	1.6361	8.0856	-11.14	-7.0601	1.2506	-1.7703	1.1572	-.3160	.5783
15	-.0373	.3773	.3182	.6545	1.3722	-.7203	.4881	-.5614	.4407	-.3463
	1.5545	-2.0993	1.6557	-7.6610	-10.89	7.2704	1.5813	1.8156	1.7166	-.4128
16	-.0285	.2356	.7536	-.5630	.1775	-.3428	.6682	-1.2603	.0715	-1.9422
	-1.4499	-1.4528	-2.5209	-1.4804	-7.8211	10.95	7.0640	-1.5133	1.4649	-1.1589
17	.0146	-.1386	.2071	-.9285	-1.2376	.6357	.0268	-.0683	-.1566	-.6838
	-1.7305	.7931	-1.7231	2.1817	-1.8111	7.5334	10.35	-7.1214	-2.5970	.0214
18	.0298	-.1849	-.4155	-.0370	-.8306	.7518	-.3698	.9597	-.0386	1.1349
	.3312	1.3569	.9980	1.5198	2.2528	1.6537	7.5166	-10.44	-7.1392	2.5484
19	.0098	.0365	-.3901	.7224	.4341	.0629	-.1625	.6370	.0955	1.2808
	1.5295	.3162	1.8487	-.4458	2.2631	-1.8361	2.8620	-7.5414	-8.1897	3.8976
20	-.0063	.1252	-.0540	.5079	.6838	-.2814	.1147	-.0500	.0831	.3232
	.8484	-.3672	.7784	-.8595	.5804	-1.5108	-.0132	-2.8304	-4.1103	3.2771

M_2

COLUMN VECTORS

ϵ_-/i	ϵ_+/i	$170\,\zeta_-$	$170\,\zeta_+$	δ_-	$B_o\epsilon_- + .170\,C_1\,\zeta_- + \delta_-$	$B_o\epsilon_+ + 170\,D_1\,\zeta_+$
-4080	4734	233	-128	-24930	-10430	-16796
8852	6526	1700	1700	- 1055	-28963	-17184
73	-914	-1296	961	4	2344	6222
3156	3293	33	72	- 334	-11212	-11171
119	-668	-531	379		- 1267	3519
1863	2001	20	43	- 204	- 6622	- 6790
137	-524	-342	242		- 1025	2584
1344	1443	14	30	- 143	- 4773	- 4901
104	-431	-253	179		- 767	2062
1054	1126	10	24	- 111	- 3744	- 3822
93	-372	-201	142		- 646	1740
858	913	9	19	- 89	- 3045	- 3101
97	-327	-167	119		- 605	1511
733	781	7	16	- 81	- 2608	- 2653
104	-304	-143	100		- 590	1370
622	687	6	14	- 65	- 2208	- 2333
129	-268	-125	88		- 648	1208
572	611	5	13	- 60	- 2032	- 2074
77	-256	-112	78		- 447	1134
550	529	6	10	- 46	- 1940	- 1799

GAMMA- K1

1		10000	629	1-	71		26	-	14		8
		-	5		4	-	3		2	-	2
2		11076-	10000-	7848	22	1568-	7-	674	5	375-	2
	-	239	2	165	-	122		93	-	74	
3		1996-	15625-	10000-	9422		2244	-	1046		611
		-	492		287	-	244		167	-	134
4		3646-	64	14126	10000	10093	-	2615		1282	
	-	774		523	-	380		287	-	222	
5		1022-	6237		13457	10000	10468	-	2856		1452
		-	898		611	-	449		344	-	277
6		2052-	36	5609	-	13079-	10000-	10692		3019	
	-	1563		990	-	692		489	-	394	
7		688-	4040		5229	-	12837-	10000-	10874		3138
		-	1662		1060	-	745		559	-	430
8		1538-	33	3662	-	5000		12692	10000	10986	
	-	3227		1739	-	1120		803	-	602	
9		554-	2980		3419	-	4833		12550	10000	11079
		-	3305		1796	-	1165		840	-	630
10		1225-	21	2751	-	3267		4707	-	12465-	10000
	-	11154		3374	-	1848		1204	-	881	
11		430-	2364		2579	-	3128		4608	-	12393
	-	10000-	11223		3415	-	1886		1242	-	907
12		1024-	26	2206	-	2469		3047	-	4544	
		12345	10000	11269	-	3441		1917	-	1287	
13		372-	1977		2092	-	2378		2980	-	4499
			12292	10000	11289	-	3496		1948	-	1289
14		870		1863	-	1988		2298	-	2919	
		4441	-	12236-	10000-	11335		3509	-	1988	
15		334-	1705		1772	-	1939		2240	-	2875
			4380	-	12203-	10000-	11368		3544	-	2006
16		716-	36	1540	-	1683		1862	-	2185	
		2830	-	4370		12178	10000	11390	-	3582	
17		267-	1490		1528	-	1642		1833	-	2139
			2788	-	4316		12147	10000	11421	-	3591
18		690		1420	-	1461		1583	-	1786	
		2110	-	2760		4302	-	12135-	10000-	11445	
19		258-	1289		1332	-	1418		1547	-	1762
			2106	-	2750		4297	-	12119-	10000-	11474
20		590		1270	-	1316		1361	-	1497	
		1724	-	2042		2722	-	4265		12114	10000

```
                      3         -        2                 1         -         1                1
 2    -    457-  10000     7085      96-   1415-       35      607       17-     337-       11
          216       7-     148-        4     109        4-      85-        2       65        2
 3       1218   14907-   10000      9443      5-     2261        5      1052        -      614
                  405        -       290               214        -      165                133
 4         14-    314-   14076    10000-  10057-       14     2605        7-     1278
          771       -      521               378        -      286               221
 5        611    5922-     10-     13429   10000-   10478              2856         -     1452
                  898        -       611               449        -      344                227
 6         36-    191-    5602        24   13063-   10000    10679        -      3016
         1561       -      989               691        -      513               393
 7        415    3811-     14-      5215            12837-   10000    10874         -     3138
                 1662        -      1060               745        -      559                430
 8         50-    134-    3662        17    5000        -    12692    10000-    10986
         3227       -     1739              1120        -      803               602
 9        305    2824        -      3397               4828        -    12538    10000-   11069
                 3302        -      1794               1164        -      840                630
10         43-    108-    2753              3269        -     4710              12473-   10000
        11161       -     3376              1849        -     1204               882
11        239    2249        -      2560               3134        -     4617              12416
  -     10000   11244        -      3421               1890        -     1244                909
12         52-     79-    2205              2467        -     3045              4541
  -     12336   10000-   11286              3434        -     1916              1286
13        201    1862        -      2092               2378        -     2980              4499
            -   12292    10000-    11289               3496        -     1948              1289
14         31-     62-    1863              1988        -     2298              2919
  -      4441            12236-    10000   11335        -     3509              1988
15        167    1605        -      1739               1940        -     2241              2876
            -    4381              12207-  10000     11371        -     3545              2007
16         72-     72-    1613              1685        -     1864              2186
  -      2832             4373        -    12186    10000-   11398              3584
17        229    1412        -      1527               1641        -     1832              2137
            -    2786               4313        -    12137    10000-   11412              3588
18         41-     41-    1423              1463        -     1585              1789
  -      2114             2764        -     4309             12154-   10000     11463
19        214    1245        -      1330               1416        -     1545              1760
            -    2103               2747        -     4292             12103-   10000     11459
20         45-     45-    1273              1318        -     1364              1500
  -      1727             1864        -     2727              4273        -    12136     10000
```

GAMMA - K1

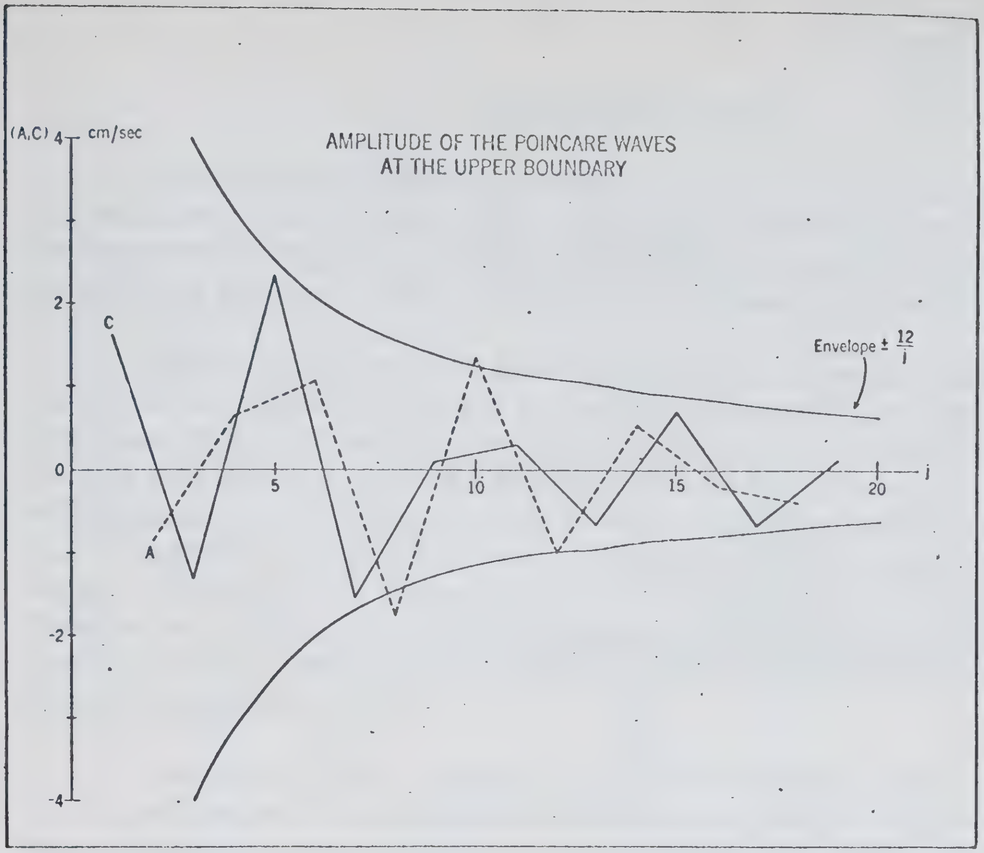
1	.9966	.0293	.0243	.0017	.0130	-.0011	.0087	-.0019	.0066	-.0023
	.0053	-.0025	.0047	-.0030	.0042	-.0034	.0039	-.0045	.0041	-.0096
2	-.1438	1.3840	1.1470	.0400	.5226	-.0538	.3452	-.0805	.2599	-.0930
	.2091	-.0982	.1837	-.1182	.1623	-.1348	.1550	-.1777	.1626	-.3758
3	-.0964	2.4177	1.2255	-.2055	.4398	-.0789	.3011	-.0815	.2298	-.0875
	.1863	-.0901	.1647	-.1076	.1458	-.1223	.1396	-.1607	.1466	-.3398
4	.0180	-.1368	.3016	-.3864	-.6059	.0489	-.2508	.0554	-.1691	.0588
	-.1302	.0601	-.1118	.0709	-.0971	.0798	-.0922	.1045	-.0968	.2210
5	.0975	-2.2081	-.8825	-.8078	-.9072	.3919	-.3435	.1451	-.2651	.1201
	-.2162	.1135	-.1920	.1303	-.1704	.1454	-.1639	.1895	-.1734	.4003
6	.0095	-.3083	-.2074	-.0879	-.4965	.3539	.4858	-.0502	.1661	-.0462
	.1030	-.0439	.0790	-.0486	.0648	-.0530	.0587	-.0676	.0587	-.1382
7	-.0970	2.1702	.8983	.4971	.5103	.5897	.7722	-.4673	.2785	-.1850
	.2284	-.1436	.2036	-.1507	.1811	-.1620	.1746	-.2071	.1849	-.4326
8	-.0256	.6130	.2940	.1325	.2592	.0704	.5487	-.3645	-.4432	.0391
	-.1290	.0336	-.0730	.0337	-.0507	.0350	-.0405	.0423	-.0370	.0827
9	.0982	-2.1745	-.9126	-.4464	-.5247	-.2721	-.3711	-.5089	-.7032	.5083
	-.2386	.2091	-.2135	.1814	-.1908	.1828	-.1839	.2249	-.1961	.4629
10	.0396	-.8996	-.4011	-.1803	-.2751	-.0844	-.2813	-.0509	-.5756	.3780
	.4155	-.0255	.1055	-.0217	.0492	-.0190	.0291	-.0208	.0180	-.0323
11	-.0990	2.1974	.9297	.4307	.5375	.2132	.3822	.1879	.2997	.4649
	.6594	-.5327	.2229	-.2486	.1992	-.2133	.1939	-.2483	.2065	-.4952
12	-.0521	1.1869	.5171	.2299	.3260	.1002	.2733	.0543	.2958	.0315
	.5915	-.3920	-.3962	.0111	-.0844	.0058	-.0280	0.0000	-.0044	-.0155
13	.1038	-2.2661	-.9635	-.4342	-.5593	-.1941	-.3993	-.1274	-.3142	-.1430
	-.2606	-.4393	-.6480	.5741	-.2116	.2837	-.2065	.2856	-.2214	.5405
14	.0706	-1.5759	-.6815	-.2990	-.4124	-.1247	-.3195	-.0621	-.2893	-.0312
	-.3168	-.0151	-.6196	.4128	.3704	.0096	.0550	.0237	-.0084	.0749
15	-.1063	2.3722	1.0108	.4483	.5892	.1898	.4219	.1054	.3337	.0799
	.2774	.1148	.2515	.3949	.6409	-.6143	.2233	-.3638	.2402	-.6081
16	-.0902	2.1110	.9077	.3969	.5418	.1605	.4036	.0759	.3417	.0322
	.3176	.0103	.3555	-.0120	.6576	-.4419	-.3258	-.0550	.0002	-.1537
17	.1236	-2.5888	-1.1069	-.4838	-.6459	-.1974	-.4644	-.0971	-.3673	-.0557
	-.3084	-.0495	-.2808	-.0624	-.2550	-.3458	-.6644	.7095	-.2726	.7393
18	.1457	-3.1663	-1.3596	-.5899	-.8027	-.2339	-.5871	-.1048	-.4785	-.0405
	-.4206	-.0043	-.4079	.0332	-.4424	.0629	-.7555	.5253	.2071	.3056
19	-.1444	3.0523	1.3068	.5671	.7647	.2268	.5526	.1024	.4401	.0424
	.3691	.0211	.3409	-.0200	.3084	-.0021	.3065	.2183	.7487	-1.1942
20	-.3519	7.4865	3.2115	1.3878	1.8881	.5437	1.3708	.2335	1.1013	.0752
	.9380	-.0098	.8685	-.1190	.8270	-.1969	.8803	-.3417	1.2645	-1.3434

	[illegible]	[illegible]	[illegible]	.0083	.0091	.0075	.0088	.0101	.0090	.0212
2	-.7664	1.0219	-.9574	.0234	-.4294	-.0491	-.2819	-.0699	-.2110	-.0795
	-.1671	-.0820	-.1516	-.0960	-.1326	-.1104	-.1286	-.1474	-.1330	-.3104
3	1.6861	-1.9191	1.0014	.2219	.3312	.0723	.2274	.0682	.1735	.0710
	.1380	.0704	.1277	.0823	.1114	.0938	.1088	.1252	.1125	.2631
4	.1317	-.0847	-.3265	-.3792	.5914	.0500	.2409	.0549	.1617	.0578
	.1257	.0614	.1032	.0664	.0922	.0767	.0862	.1000	.0898	.2098
5	-1.7626	1.7300	-.6662	.7848	-.8004	-.3869	-.2709	-.1332	-.2095	-.1048
	-.1697	-.0965	-.1534	-.1053	-.1361	-.1184	-.1318	-.1545	-.1377	-.3242
6	.2326	-.2356	.1761	-.0833	.4813	.3539	-.4967	-.0513	-.1744	-.0481
	-.1108	-.0478	-.0831	-.0496	-.0704	-.0556	-.0640	-.0722	-.0649	-.1491
7	1.7684	-1.7121	.6894	-.4787	.4079	-.5949	.7025	.4546	.2255	.1695
	.1840	.1266	.1669	.1253	.1488	.1342	.1456	.1741	.1521	.3602
8	-.4978	.4820	-.2337	.1283	-.2310	.0721	-.5292	-.3606	.4575	.0436
	.1422	.0402	.0816	.0387	.0600	.0420	.0488	.0512	.0464	.1026
9	-1.7825	1.7151	-.7044	.4269	-.4214	.2774	-.3012	.5211	-.6495	-.4931
	-.1939	-.1924	-.1764	-.1557	-.1584	-.1552	-.1545	-.1914	-.1631	-.3901
10	.7367	-.7096	.3147	-.1732	.2333	-.0867	.2527	-.0560	.5533	.3716
	-.4353	-.0348	-.1189	-.0301	-.0627	-.0297	-.0412	-.0341	-.0316	-.0619
11	1.8066	-1.7314	.7181	-.4109	.4327	-.2185	.3113	-.2003	.2455	-.4810
	.6154	.5165	.1849	.2222	.1664	.1855	.1641	.2143	.1731	.4214
12	-.9793	.9385	-.4045	.2204	-.2704	.1037	-.2357	.0615	-.2669	.0409
	-.5686	-.3814	.4145	.0229	.1019	.0199	.0437	.0177	.0220	.0232
13	-1.8719	1.7899	-.7473	.4148	-.4531	.1998	-.3274	.1402	-.2593	.1592
	-.2149	.4560	-.6098	-.5481	-.1781	-.2555	-.1761	-.2511	-.1874	-.4654
14	1.3036	-1.2465	.5312	-.2873	.3400	-.1292	.2702	-.0714	.2515	-.0432
	.2856	-.0291	.5955	.3973	-.3936	-.0090	-.0757	.0004	-.0147	.0238
15	1.9642	-1.8751	.7853	-.4265	.4771	-.1955	.3462	-.1186	.2759	-.0965
	.2293	-.1320	.2108	-.4231	.6060	.5848	.1912	.3276	.2043	.5291
16	-1.7530	1.6742	-.7091	.3794	-.4431	.1667	-.3371	.0885	-.2909	.0482
	-.2755	.0289	-.3227	.0095	-.6269	-.4169	.3534	-.0236	.0308	-.0850
17	-2.1273	2.0311	-.8536	.4615	-.5220	.2036	-.3804	.1117	-.3032	.0738
	-.2549	.0678	-.2350	.0943	-.2158	.3796	-.6288	-.6696	-.2326	-.6513
18	2.6226	-2.5020	1.0568	-.5648	.6543	-.2430	.4868	-.1237	.4020	-.0643
	.3568	-.0316	.3578	.0001	.3958	.0247	.7128	.4779	-.2540	.2014
19	2.5068	-2.3902	1.0074	-.5375	.6173	-.2326	.4526	-.1183	.3636	-.0625
	.3052	-.0397	.2836	-.0203	.2615	-.0384	.2637	-.2682	.7011	1.0894
20	-6.1877	5.9009	-2.4921	1.3240	-1.5352	.5625	-1.1317	.2761	-.9186	.1289
	-.7855	.0490	-.7438	-.0352	-.7154	-.1042	-.7786	-.2271	-1.1504	-1.0940

K_1

COLUMN VECTORS

ϵ_-/i	ϵ_+/i	δ_-
− 823	−1871	−38495
18025	11594	− 1167
6103	4154	− 1189
3675	2503	− 716
2609	1773	− 518
2364	1393	− 387
1944	1129	− 315
1677	932	− 280
1468	1290	− 251
1339	1098	− 203
1180	1000	− 181


AMPLITUDE OF THE POINCARE WAVES
AT THE UPPER BOUNDARY
(A,C)
cm/sec
4
2
0
-2
-4
5
10
15
20
j
C
A
Envelope ± 12/j

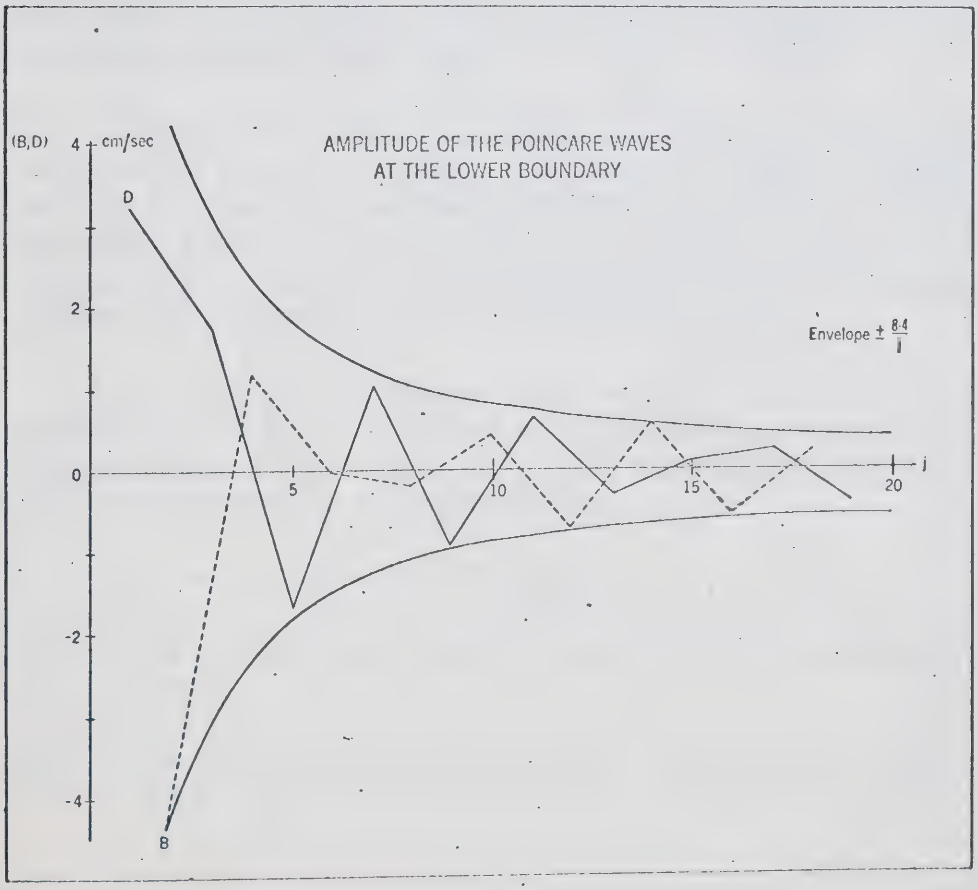

AMPLITUDE OF THE POINCARE WAVES
AT THE LOWER BOUNDARY
(B,D)
4
cm/sec
2
0
-2
-4
5
10
15
20
j
D
B
Envelope ± 8.4/j

Fig. 10

The Solution for M_2.

Following the procedure outlined on p. 76 ff., we obtain the values of the arbitrary constants (A_n, C_m), (B_n, D_m) in 2 for M_2; their variation with j is illustrated in Fig. 10. It can be noticed that these quantities decrease at least as $(-)^j/j$ if not more rapidly. (The constant chosen for the envelope is arbitrary).

No such convergence is to be found for K_1. If we inspect Γ_+^{-1} and Γ_-^{-1} for K_1 on p. 115 and 116 we note that the second element of each row increases with j (alternate rows). Since this element is multiplied by the second element of the column vector ϵ on p. 117 (this element being by far the most important element of the column vector on the right) in order to obtain the further constants, once B_0 is evaluated, it will have as a consequence that the Poincaré waves become larger and larger as j increases so that v will oscillate violently at the inner boundaries. This means that a convergent solution for K_1 does not exist. The reasons for this disappointing conclusion will be investigated once we have carried through our solution for M_2.

Going back to M_2, we notice that if the Poincaré waves vary as $(-)^j/j$ in 2, this will cause v to diverge logarithmically at $(\pm a_2, \pm l_2)$. In fact, using the 20 known constants, the plot of v at $\pm l_2$ (not illustrated) shows that it has large oscillations in the vicinity of $\pm a_2$ and we may surmise that these oscillations are the indication of such a type of divergence. Z on the other hand is finite and smooth everywhere.

This situation was to be expected considering the sharply discontinuous character of the inner corners. This however violates the condition we had set on p. 57 for the representation of discontinuous functions. $V(\pm l_2)$ can only represent a function which has a finite discontinuity at $\pm a$. It makes it then impossible, strictly speaking, to find solutions in 1 and 3 from those in 2: unless we find a way of removing this infinite discontinuity.

If we go back to equations (43) and (44) which describe the boundary condition on v at $\pm l_2$, we notice that the Kelvin waves in 1 and 3 are determined independently of the Poincaré waves in 2; they depend exclusively on $A_0^{(2)}$ and $B_0^{(2)}$. This provides a clue to the solution.

We are given the following facts:

1) The Poincaré waves only are responsible for the divergence at the corners.

2) The Kelvin waves are determined everywhere once they are known in 2, divergence or not.

Table VIII

SEA	1		2		3		
J	(A, C)	(B, D)	(A, C)	(B, D)	(A, C)	(B, D)	
0	-1.26	.27	-1.52	3.46	1.74	.00	
1	.29		1.61	3.18	- .49	.21	
2	.40		- .89	-4.21	.00	- .77	
3	- .29		-1.33	1.76	.06	.39	
4	.09		.65	1.15	.00	.39	
5	- .12		2.40	-1.67	- .02	- .19	
6	- .07		1.07	- .01	.00	- .29	
7	.02		-1.56	1.05	- .01	.11	
8	- .03		-1.80	- .19	.00	.15	
9	.07		.10	- .90	.01	- .05	
10	.05		1.35	.45	.00	- .02	
11	- .01		.33	.68	.00	- .01	
12	.03		- .98	- .69	.00	- .07	
13	- .04		- .61	- .28	.00	.05	
14	- .03		.58	.61	.00	.12	
15	.00		.71	.14	.00	- .06	
16	- .03		- .18	- .50	.00	- .11	
17	.04		- .65	.33	.00	.05	
18	.01		- .25	.30	.00	.07	
19	.01		.14	- .35	.00	.02	CM/SEC

We know as well that the singularity is to be found at $\pm\ l_2$ and not within 2 due to presence of the exponential damping factors.

We use this damping feature in the following way: we equate hv at l_1 and $-l_3$ to a hv in 2 made up of the superposition of the Kelvin waves at the inner boundaries $\pm l_2$ and the contribution of the Poincaré waves at some point within 2.

In this way we moderate the effect of the corners in 1 and 3 and we may obtain solutions for the two seas. This makes physical sense since infinite velocities do not exist and equating infinite transports only leads to indeterminate solutions.

The only element of arbitrariness is the distance we choose within 2 to pick the Poincaré waves; this will have as a consequence that the solution for v and Z in the immediate vicinity of $+l_1$ and $-l_3$ will vary slightly with our choice. But this indeterminacy does not travel very far in 1 and 3 where the solution rapidly becomes unique.

The point we choose where to evaluate the contribution of the Poincaré waves to hv is $y_2 = 0$; this is an obvious choice since it lies halfway between $\pm\ l_2$ and sea 2 is very short compared to 1 and 3.

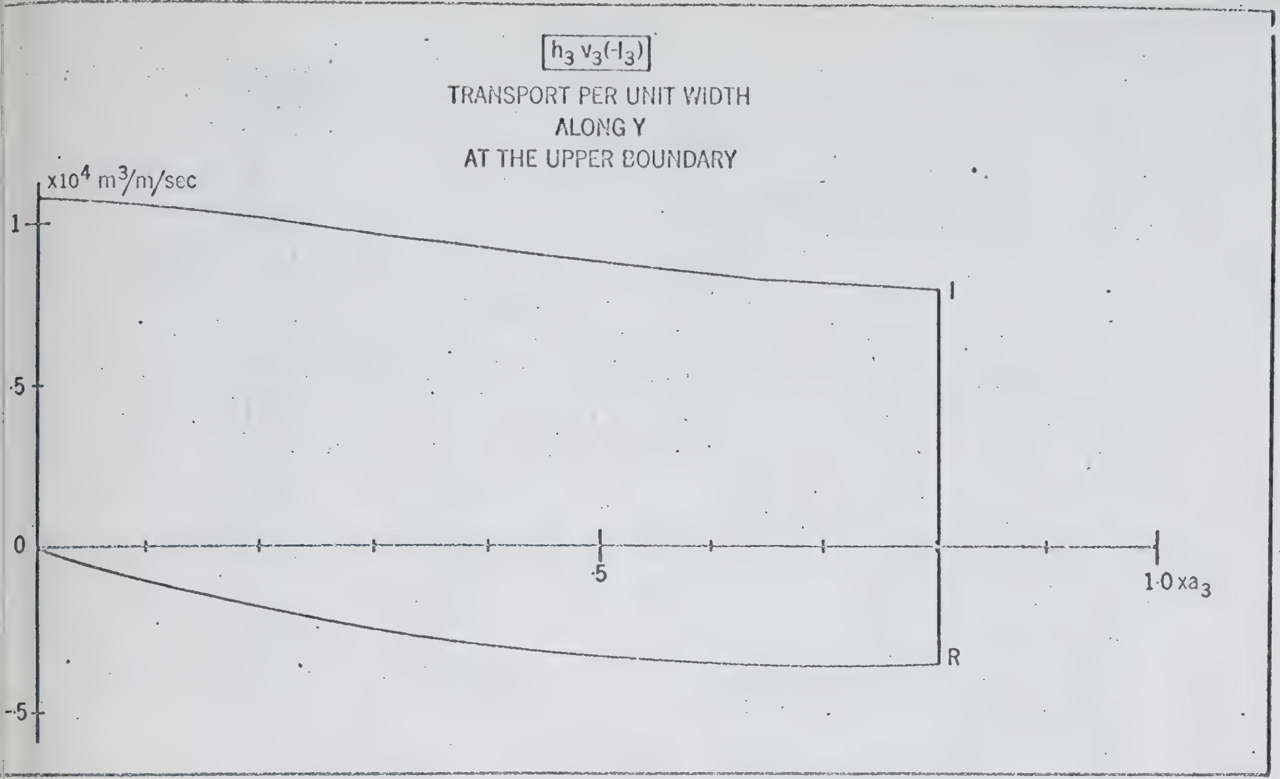
In this way we obtain a complete set of values for $(A_n^{①}, C_m^{①})$, $(B_n^{③}, D_m^{③})$ which are tabulated in Table 8 along with the other constants.

The transports per unit width and the elevations at the inner boundaries are shown in Fig. 11 and 12. Some discontinuity will be noticed in the elevations since we have contradicted the solution obtained in 2 where we had assumed the Poincaré waves in 2 to be balanced at $\pm\ l_2$. These discrepancies are slight and could be removed completely if the computations were repeated taking into account the modifications introduced into the boundary condition on v. There are no discrepancies in the transports since $(A_n^{①}, C_m^{①})$ and $(B_n^{③}, D_m^{③})$ have been chosen to satisfy the modified boundary condition exactly.

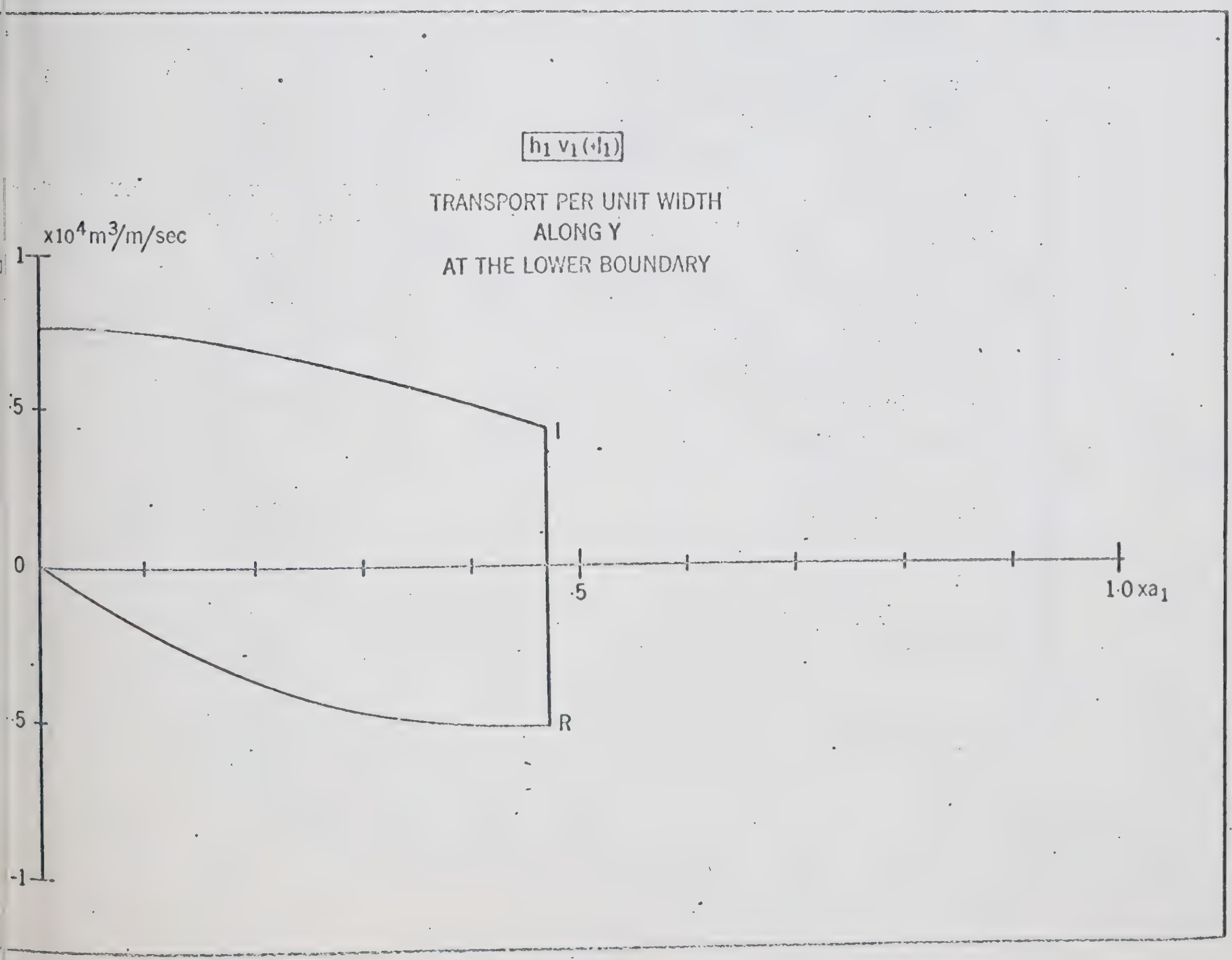
Charts showing Z, hv and the elements of the current ellipses for M_2 are given in Fig. 13 and 14.

Appendix 2 Part 3 gives a comparison between the transport bhv ($Z_o = 70$cm) for the 6 channel model and that for the 3 basin model.

The solution given in Table 8 has been multiplied by the factor 7.5 in order that the value of Z_o be approximately 70 cm across the mouth.

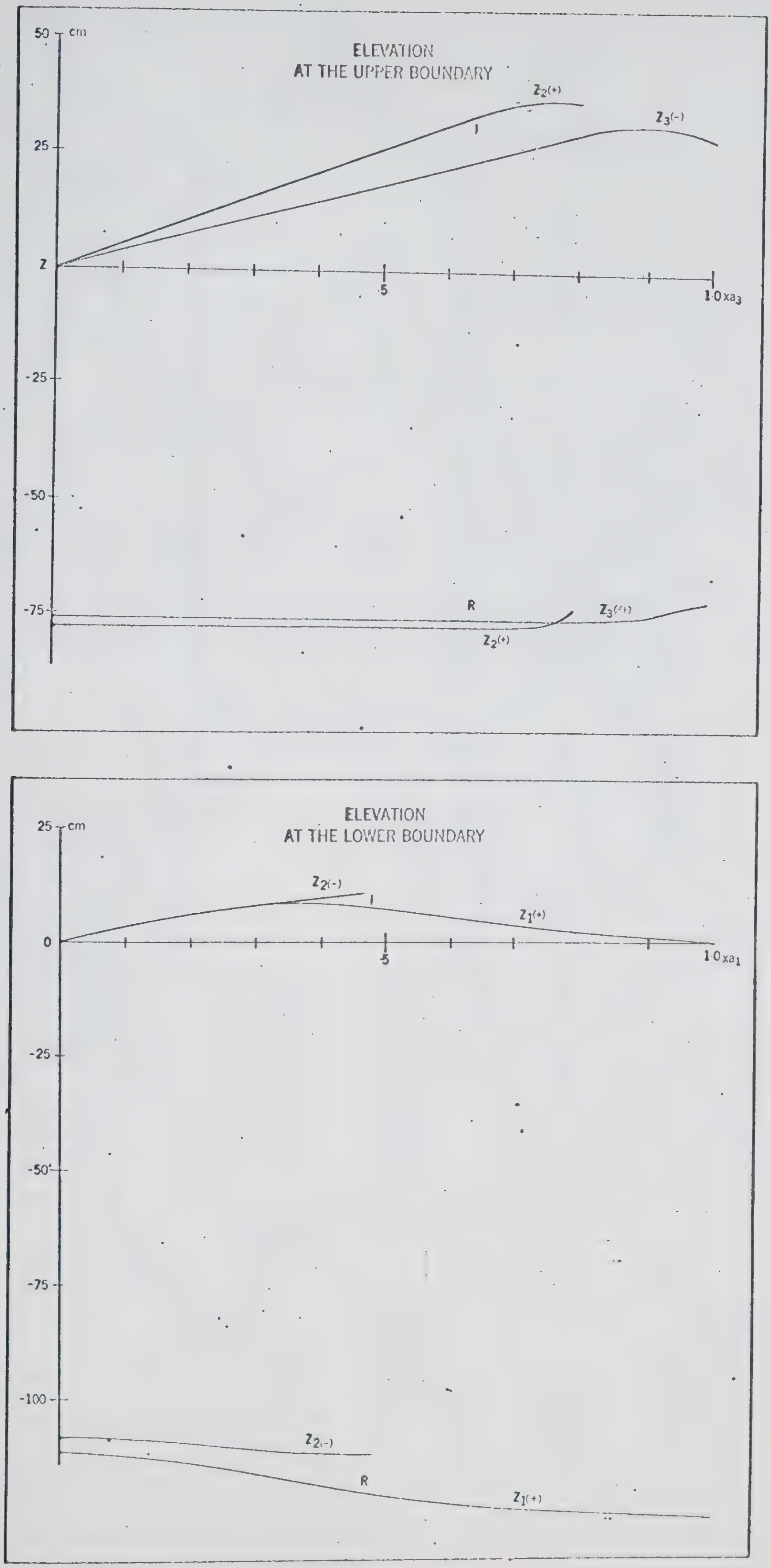


$h_3 v_3(-l_3)$
TRANSPORT PER UNIT WIDTH
ALONG Y
AT THE UPPER BOUNDARY
x10^4 m^3/m/sec
1
·5
0
-·5
·5
1·0 xa_3
I
R



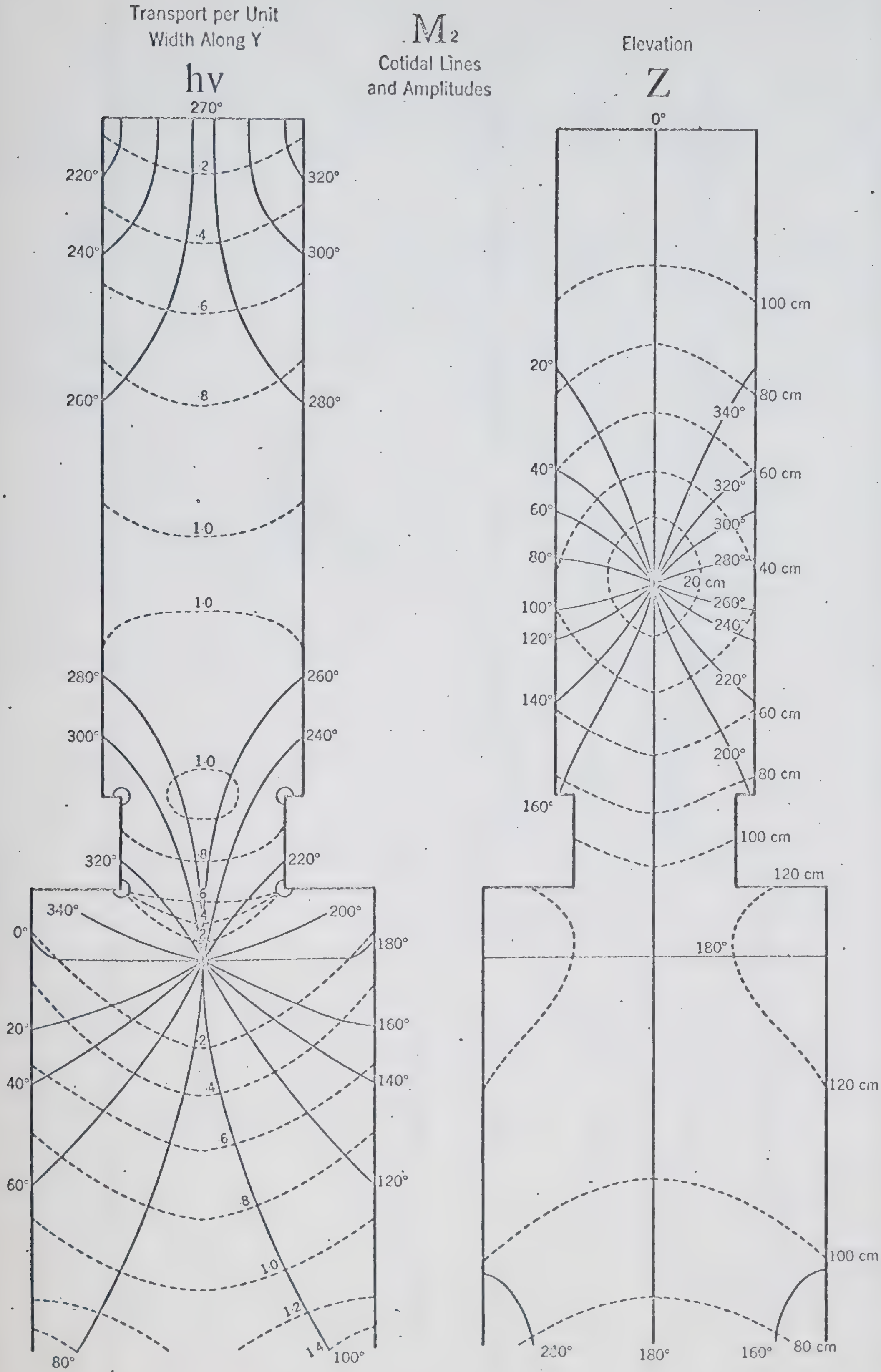
$h_1 v_1(+l_1)$
TRANSPORT PER UNIT WIDTH
ALONG Y
AT THE LOWER BOUNDARY
x10^4 m^3/m/sec
1
·5
0
-·5
-1
·5
1·0 xa_1
I
R

Fig. 11



ELEVATION
AT THE UPPER BOUNDARY
50
cm
25
Z
-25
-50
-75
·5
1·0 xa3
ELEVATION
AT THE LOWER BOUNDARY
25
cm
0
-25
-50
-75
-100
·5
1·0 xa1
R

Fig. 12



Transport per Unit Width Along Y
hv
M2
Cotidal Lines and Amplitudes
Elevation
Z
270°
220°
320°
240°
300°
260°
280°
1·0
280°
260°
300°
240°
320°
220°
340°
200°
0°
180°
20°
160°
40°
140°
60°
120°
80°
100°
0°
100 cm
20°
80 cm
340°
40°
60 cm
320°
60°
300°
80°
280°
40 cm
20 cm
100°
260°
240°
120°
220°
140°
60 cm
200°
80 cm
160°
100 cm
120 cm
180°
120 cm
100 cm
200°
180°
160°
80 cm

Fig. 13

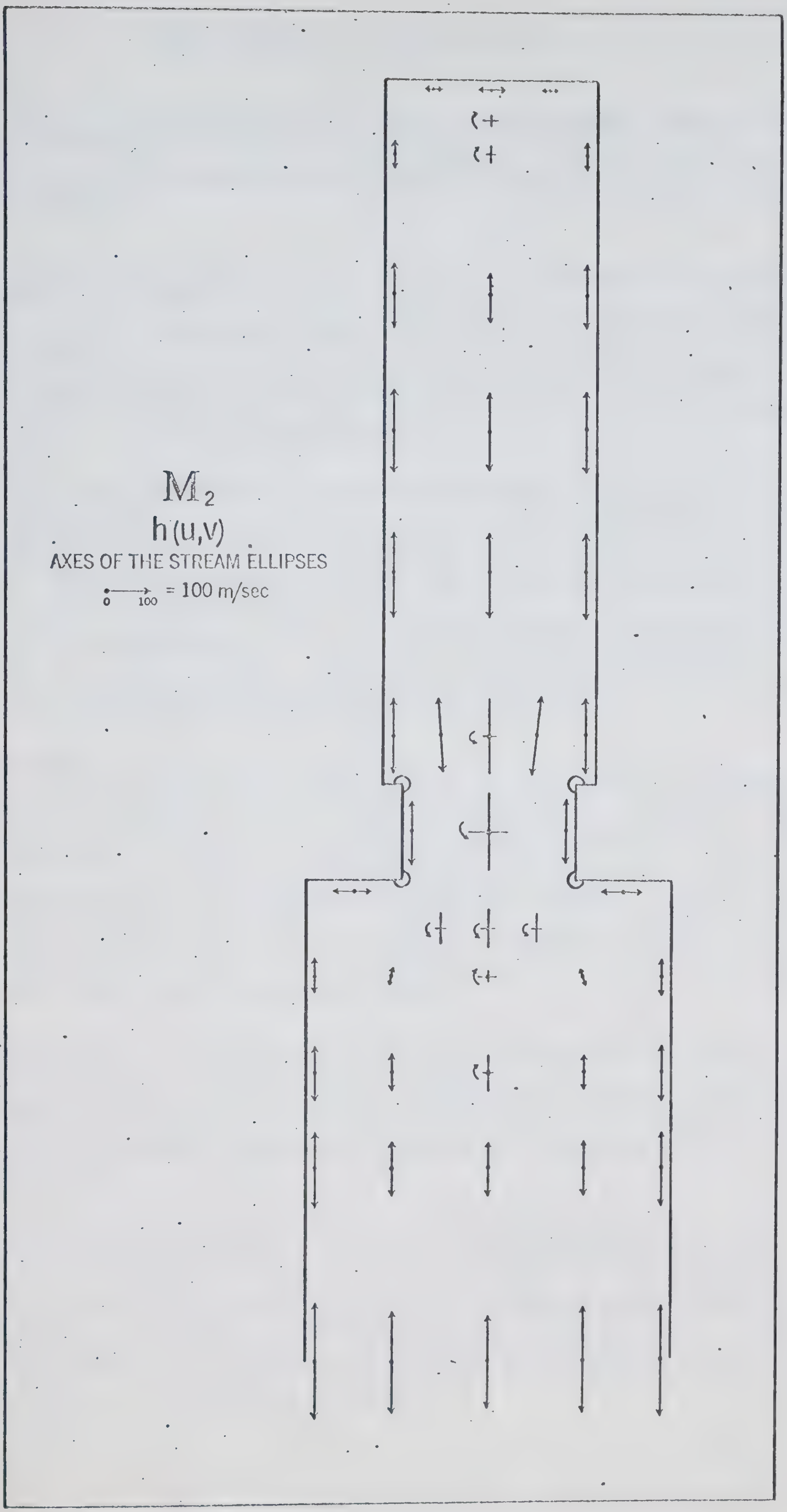


Fig. 14

Discussion of the Results for M_2

Once again theory supplies us with more information than that given by observations. Not only do we obtain a cotidal chart for the elevation but as well a cotidal chart for hv (the most important part of the transport) and a complete map of the tidal ellipses.

If we compare the cotidal chart for Z with the one based on observations (Fig. 2) we notice a remarkable qualitative agreement between the two charts. This agreement is most remarkable when we consider the crudeness of the model used to represent the sea. As a matter of fact the points of disagreement between the two charts indicate the effects of the features not taken into account in the model and to be found in the actual sea.

The 160° line in the model curving on the Greenland coast has its counterpart in the empirical chart. The 180° line is seen in the model to go straight up to the node while another 180° line goes across the Sea of Labrador just ahead of Davis Strait. Coastal observations allow the two same 180° lines to be drawn in the empirical chart. The 200° line in the theoretical chart has its counterpart in the empirical chart but there we see that nearer the coast it is followed by a 220° and a 240° line. These indicate the effect of the important shelf lying off the Labrador coast which had been neglected in the model.

In Baffin Bay, the theoretical chart puts the 90° and 270° lines straight across while in the empirical chart they are in the shape of an inverted V. More, in the empirical chart the cotidal lines are more crowded in the lower part of the Bay. These two features reflect the fact that in the model the depth in the Bay decreases discontinuously at its junction with Davis Strait while in actuality the change is gradual and causes some slowing down in the progress round the node in this area. The position of the 000° line underlines the asymmetry in the basin which has not been taken into account in the model.

When it comes to the amplitudes it can be noticed that the two 120 cm lines in the calculated chart have their counterpart in the observed chart. However there is in reality a considerable amplification of the tide on the Canadian side of the sea which the model cannot reproduce. The overall agreement in the amplitudes is quite good.

We cannot compliment ourselves on the position of the node since the length of sea 3 has been chosen in such a way that it would fall where expected. The last 300 km of sea 3 in the model are fictitious and aim at representing the effect of Smith Sound. It should be noted that in the actual sea the node is closer to Baffinland; the bathymetric chart shows that it lies approximately where the greatest depth across the section is to be found.

The Divergence at the Inner Boundaries for K_1

It does not seem easy at first sight to unravel from the complicated expressions we have been handling the reason why the solution for K_1 diverges. However a bit of analysis will reveal that it is essentially related to the fact that

$$\sigma < 2\omega$$

for K_1 due to the high latitude of the sea under study.

If we go back to equations (48) and look at the order of magnitude of the terms as $j \to \infty$ we notice first that the interaction terms already tabulated in Tables 5 and 6 decrease at least as $1/j^3$ and can therefore be neglected in comparison with terms decreasing as $1/j$ or $1/j^2$.

If we choose to look at the equation corresponding to $k = 3$ (the simplest to handle: the reasoning would have been identical had we chosen $k = 1$) we obtain expressions of the form

$$\alpha_2 \left[\sigma \frac{A_n^{(2)}}{K_n} - 2\omega \sum \frac{P_{mn}}{m} C_m^{(2)} \right] \simeq A_o^{(3)} \left(\frac{h_3}{g}\right)^{\frac{1}{2}} u_n^{(3)} \cos 2K_o^{(3)} l_3 - \left(\frac{h_2}{g}\right)^{\frac{1}{2}} p_n^{(2)} A_o^{(2)}$$

$$i \alpha_2 \left[\sigma \frac{C_m^{(2)}}{K_m} - 2\omega \sum \frac{Q_{nm}}{n} A_n^{(2)} \right] \simeq B_o^{(2)} q_m^{(2)} \left(\frac{h_2}{g}\right)^{\frac{1}{2}} - i A_o^{(3)} \left(\frac{h_3}{g}\right)^{\frac{1}{2}} V_m^{(3)} \sin 2K_o^{(3)} l_3$$

as $j \longrightarrow \infty$

if we care to remember the definitions of $G^{(3)}$ and $A_o^{(3)}$.

These equations simply state that the difference between the Kelvin waves in 2 and 3 should be balanced by the Poincaré waves in 2 when $j \gg 1$. This makes physical sense since the Kelvin waves are the main mode of motion and that the presence of a boundary will create Poincaré waves whose only purpose is to balance the Kelvin waves and thus satisfy the boundary condition. Also we have already noticed in the solution of M_2 that the Poincaré waves in 1 and 3 diminish much more rapidly than the Poincaré waves in 2.

We notice next that

$$p_n, u_n, q_m, v_m \text{ decrease as } 1/j^2$$

while

$$1/K_j \sim 1/j$$

and $|P_{mn}|$ /m, $|Q_{nm}|$ /n $\sim$ $2/\pi\, j$ for $m = n \pm 1$, $n = m \pm 1$
$j >> 1$

(59) is then equivalent to the relations

$$\ldots A_{m-1} + C_{m-2} + A_{m+1} \ldots = \text{const}/m$$

$$\ldots C_{n-1} + A_{n-} + C_{n+1} \ldots = \text{const}/n \qquad (60)$$

The further terms on the left and the right indicated by dots are affected by the factors 1/3, 1/5, 1/7 etc.

If we inspect the 3 main elements of each row of the matrices $\Gamma_\pm$ for K_1 on p. 115, 116 we note that the factor affecting the constant in $j - 1$ is consistently larger than the one affecting the constant in $j + 1$ but that as $j \rightarrow \infty$ both tend to become equal and in constant ratio to the diagonal element. This ratio gives the value of $(2/\pi) / (\sigma/2\omega)$. The column vector (renormalized) on p. 119 decreases as $1/j$.

Now suppose we solve a set of 6 equations of the form (60) by successive elimination from below, assuming j to be so large that the right hand side is approximately zero, so that all the unknowns are expressed in terms of the one of smallest order. We obtain two distinct sets of values for the unknowns depending on whether the ratio $r = (\sigma/2\omega) / (2/\pi)$ is smaller or larger than 1.

Taking A_j as the unknown of lowest order and trunkating C_{j+7} away (This is always so: the term just to the right of the last diagonal element is always removed when we give finite dimensions to the matrices), we obtain:

		$r << 1$	$r >> 1$
A	=	A_j	A_j
C_{j+1}	=	$r A_j$	A_j/r
A_{j+2}	=	A_j	A_j/r^2
C_{j+3}	=	0	$-A_j/r^3$
A_{j+4}	=	$-A_j$	A_j/r^4
C_{j+5}	=	$-rA_j$	$-A_j/r^5$
A_{j+6}	=	$-A_j$	$-A_j/r^6$

It can be seen that for $r << 1$ the unknowns just oscillate around A_j while for $r >> 1$ the convergence gets stronger and stronger as we move into the higher orders.

The above calculations are very rough but indicate the crucial role played by r. The criterion for convergence seems to be

$$\sigma/2\omega > 2/\pi = .64$$

and a more stringent form would be

$$\sigma/2\omega > 1$$

There is no way of checking this analytically due to the great algebraic complication. But even if we take the criterion for convergence in its weakest form there is little chance of getting a convergent solution for any diurnal constituent. For OO_1 the fastest diurnal constituents (σ = 16.1°/hour),

$$\sigma/2\omega = .60 < .64$$

Unless we choose to bring the sea to a more southern latitude no convergent solution exists for the diurnal tides.

The above considerations give the mathematical reason for the divergence. There is no obvious physical interpretation for such a state of things.

The source of the difficulty can be traced to (27) expressing Z in a form which is such that $u(\pm a) = 0$ is satisfied.

Had we written Z directly in terms of the basic vectors introducing a new set of arbitrary constants affecting these vectors we could have evaluated these constants easily from the boundary condition on the elevations.

On the other hand writing the solution for Z in such a form would result in an expression for u which would violate the above mentioned boundary condition. Since the whole formulation of our problem rests on the logical consequences deriving from the satisfaction of this boundary condition (eigenvalues, eigenfunctions, basic sets), we cannot throw all this away just for the sake of getting an answer.

This leaves us without any solution for K_1 but from the results obtained for M_2, we may make some conjectures about it.

First the one dimensional calculations give a very good clue about the two dimensional solution once we have translated the trigonometric waves into Kelvin waves. For instance, for M_2, the position of the node found in one dimension is the same as the one found in two dimensions and the profile of elevation along the central line of the sea is given closely by the one dimensional profile except in and around Davis Strait.

For K_1 we may assume that we would have obtained a node in the Strait, relatively small elevations in 1 and larger elevations in 3.

This however would not compare well with Fig. 3 where the node lies on Cumberland Peninsula. This brings to our attention the fact that the model we chose is completely symmetrical in x while it is obvious that the actual sea does not exhibit such complete symmetry; sea 1 and 3 would have been more representative if we had brought them more to the left of the central line of symmetry in 2.

This asymmetry does not seem to affect M_2 appreciably since nothing crucial happens to this constituent in 2. However for K_1 such an asymmetry should have a marked influence on the position of the node and the projection of

Cumberland Peninsula into the Strait must be responsible for the disturbance in the position of the point of amphidromy.

GENERAL CONCLUSIONS

We have investigated with the help of theory the tidal regime in the sea made up of Baffin Bay, Davis Strait and the Labrador Sea.

Our study has shown that the tides are induced mainly by those in the Atlantic; also the motion seems to be of a standing wave character which is interpreted as the perfect reflection of a travelling wave at the head of the sea.

The shape of the basin suggested immediately that linear motion should predominate. Accordingly one dimensional calculations were carried out which supported well the general variation in amplitude along the sea, once proper account was taken of the diffuse reflection which occurs at the head.

A by-product of this investigation was values for the mean currents across the sea. Such values have been confirmed by two dimensional calculations for M_2 and therefore give some indication on the horizontal motion prevailing in such a sea.

Two dimensional calculations over the same basin proved to be exceedingly difficult in spite of the drastic schematization of the width and depth and lead to satisfactory results only when $\sigma < 2\omega$, otherwise divergence occurs at the inner boundaries which makes absurd any attempt at satisfying inner boundary conditions. When convergence occured however the results were most rewarding and redeemed the effort spent.

For the semidiurnal tide M_2, we conclude that:

1) There is a maximum amplitude in elevation just ahead of Davis Strait,

2) A node is found at latitude 70°N nearer to Baffinland than to Greenland,

3) High water is simultaneous along the center of the sea till one reaches the node where there is a change of phase of 180°,

4) The tide progresses very slowly round the node in the southern section of Baffin Bay,

5) The very large tides experienced in the vicinity of Hudson Strait are most likely prevalent only in the immediate vicinity of the coast,

6) The currents are alternating over most of the sea; marked elliptic currents are to be found on the Labrador Shelf, in the vicinity of Davis Strait and at the head of Baffin Bay,

7) The maximum currents over the body of the sea are found in Davis Strait and are of the order of 15 cm/sec,

8) The weakest currents are to be found at the southern opening of Davis Strait,

9) The current at the mouth of the Sea of Labrador is of the order of 10 cm/sec,

10) The gradients in the elevation and the currents across a section are due mainly to the gradients inherent to the superposition of two standing Kelvin waves of an odd and even character since the motion over the body of the sea is predominantly Kelvin.

The areas where this does not hold are

a) The Labrador Shelf where large currents and elevations are induced by the topography,

b) The entrance (64°N) and exit (67°N) of Davis Strait.

The conclusions for the diurnal tide K_1 are not so categorical due to the difficulties encountered during the calculations. We state

1) The diurnal elevations are small over Davis Strait and the Labrador Sea; they become important in the northern sections of Baffin Bay,

2) Diurnal streams are important in Davis Strait and in the northern section of the Labrador Sea; they most likely predominate over the semidiurnal streams in the latter area while they are of the same order of magnitude as the semiurnal streams in the Strait itself.

The same conclusions hold for the other diurnal and semi diurnal tides; the numerical values mentioned above should be reduced by factors representing the ratio of the amplitude of these tides to that of K_1 or M_2.

ACKNOWLEDGEMENTS

The author wishes to thank the following persons or groups of persons:

Dr. J.R. Rossiter, Director of the Tidal Institute, who has supervised the research carried out for this thesis and who, many times, has come to the writer's rescue. Special thanks for his generous permission to use all the facilities of the Institute, especially the most complete and important Library on Tides.

The Senate of the University of Liverpool, who kindly allowed the writer to carry out the latter part of his work in Ottawa at the Division of Oceanographic Research of the Government of Canada, where he could use the computational facilities of the Department of Mines & Technical Surveys.

Dr. W.M. Cameron, Director of the Marine Sciences Branch, Department of Mines & Technical Surveys, who made it possible for the author to undertake this research project, and Dr. N. J. Campbell and Mr. F.G. Barber for their sympathetic interest and encouragement in this work. Un grand merci à M.M. Isabelle pour les diagrammes.

Finally sincere thanks to all the members of the Tidal Institute who, by their constant interest and their personal friendship have helped make the writer's stay at Bidston Observatory most pleasant and fruitful.

LIST OF FIGURES AND TABLES

Fig. 11. A plot of hv, the transport per unit width along y, at the upper and lower boundaries of seas 1 and 3. These transports are exactly equal to the transport hv for sea 2 at their common junction, while they are zero outside that interval.

Fig. 12. A plot of the elevations Z at their common boundaries. These elevations are not exactly equal because the boundary conditions have been slightly violated by removing the infinite velocities are the inner corners after the matrices had been inverted. If one would remove the infinite velocities before inverting the matrices, the elevations would be equal over their common junction.

Fig. 13. The cotidal lines and the amplitude distribution of the elevation Z and of hv, the transport per unit width along y, for M_2, obtained by solving the equations as hydrodynamics for the three sea model. The solution given in Table VII has been multiplied by the factor 7.5 so that the elevation across the mouth is approximately 70 cm. The amplitude of hv is given in units of 100 $m^3/m/sec$.

Fig. 14. The elements of the current ellipses. The current is rectilinear almost everywhere. The sense of rotation of the current is indicated by the curved arrow. The double arrow indicates alternating currents.

Table I

Tidal Stations. Name, geographical location, duration of the interval of observations, authority having performed the analysis.
(C.H.S. - Canadian Hydrographic Service)

Table II

Tidal Data. Amplitude and phase for Zone +0400 of K_1, 0_1, P_1, M_2, S_2 and N_2 at the stations noted in Table I.

Table III

Data on the schematization of the Labrador Sea, Davis Strait and Baffin Bay into one or more one dimensional channels.

Table IV

Defant's calculations repeated. A comparison of Defant's 10 rectangular channel model with a 20 rectangular channel model and a 3 exponential channel model.

Table V

Terms arising from the interaction between the Kelvin and the Poincaré waves.

Table VI

Terms arising from the interaction between the Poincaré waves.

Table VII

Numerical values of the matrices and of the column vectors.

Table VIII

Solution of the equations of hydrodynamics for the three rectangular sea model, assuming the current component v to be given across the mouth by a standing Kelvin wave of amplitude -i. The solution gives the values of A_n, B_n, C_m and D_m, the amplitudes of the Kelvin and Poincaré waves in the three seas.

BIBLIOGRAPHY

1. - Al'tschuler V.M. and Vladimirov O.A. G.V. Polukarov's Method used to calculate the semidiurnal Tide (M_2) in the Baffin Sea and Baffin Strait. Trudy Gosvd. Okean. Instituto. 64, 112, 1961. Translation: National Lending Library for Science and Technology. Russian Translating Programme. RTS 2094.

2. - Defant A. Die Gezeiten des Atlantischen Ozeans und des Arktischen Meeres. Ann. Hydr. Mar. Met. 52, 153, 1924.

3. - Grace S.F. Tidal Oscillations in Rotating Rectangular Basins of Uniform Depth. MNRAS, Geophys. Suppl. 2, No. 8, 385, 1931.

4. - Green L. On the Motion of Waves in a Variable Canal of Small Depth and Width. Camb. Trans. 6, 225, 1837.

5. - Halmos P.R. Finite-Dimensional Vector Spaces. Van Nostrand Co. Princeton. 1958.

6. - Hansen W. Gezeiten und Gezeitenströme der halbtägigen Hauptmondtide M_2 in der Nordsee. Dtsch. Hydrogr. Z., Ergänzungsheft 1, 1952.

7. - Harris R.A. Manual of Tides, Part IVB. Cotidal Lines of the World, 388, 1904.

8. - Kamke E. Differentialgleichungen. Chelsea Publishing Company. 435, 1948.

9. - Lamb H. Hydrodynamics. Dover Publications. New York. 1932.

10. - McCowan J. The Theory of Long Waves and its Application to the Tidal Phenomena of Rivers and Estuaries. Phil. Mag. (5), 33, 250, 1892.

11. - Poincaré H. Leçons de Mécanique Céleste, t. 3 (Théorie des Marées), 124, 1910.

12. - Proudman J. The Tides of the Atlantic Ocean. MNRAS, 104, 244, 1944.

13. - Schureman P. A Manual of the Harmonic Analysis and Prediction of Tides. U.S. Coast and Geodetic Survey. Special Publication 98, 162, 1941.

14. - Sterneck R. Die Gezeiten der Oceane. 1. Mitteilung. S.B. Akad. Miss. Wien (Math.-Naturwis. K1.) 129, 131, 1920.

15. - Taylor G.I. Tidal Oscillations in Gulfs and Rectangular Basins. Proc. Lond. Math. Soc. (2), 20, No. 2 and 3, 148, 1921.

16. - Thomson W. On Gravitational Oscillations of Rotating Water. Proc. Roy. Soc. Edinb. 10, 92, 1879.

APPENDIX 1

SOLUTIONS OF EQUATIONS (3) AND (4)

The differential equations in Z resulting from equations (3) and (4) is

$$\frac{1}{b}\frac{d}{dy}\left[b\,h\,\frac{dZ}{dy}\right]+\frac{\sigma^2}{g}Z=0 \qquad (1-1)$$

and v is related to Z through the relation

$$v=-\frac{g}{\sigma}\frac{dZ}{dy} \qquad (1-2)$$

1.- Constant Width and Depth

The laws of width and depth are

$$b=b_o \qquad h=h_o$$

(1-1) takes the form

$$\frac{d^2Z}{dy^2}+K^2Z=0 \qquad (1-3)$$

where

$$K=\frac{\sigma}{(gh)^{\frac{1}{2}}}$$

This equation has for general solution

$$Z=A\cos Ky+B\sin Ky \qquad (1-4)$$

implying

$$v=\left(\frac{g}{h}\right)^{\frac{1}{2}}\left[A\sin Ky-B\cos Ky\right] \qquad (1-5)$$

A and B are arbitrary constants of integration; this will also be the case for the further solutions that will follow.

2. - Exponential Variation in Width and Depth

The y dependence of b and h is given by

$$b = b_o \cdot e^{\beta y} \qquad h = h_o \cdot e^{\alpha y}$$

(1-1) takes the form

$$\frac{d^2Z}{dX^2} - \frac{1}{X}\,\frac{\beta}{\alpha}\,\frac{dZ}{dX} + \frac{\sigma^2}{g h_o \alpha^2} \cdot \frac{Z}{X} = 0$$

through the substitution

$$X = e^{-\alpha y}$$

This equation is related to Bessel differential equation and it has for solution (Kamke (14))

$$Z = \left(\frac{h_o}{h}\right)^{\frac{1}{2}\left(1+\frac{\beta}{\alpha}\right)} \cdot \left[A\, J_{\left[1+\frac{\beta}{\alpha}\right]}(x) + B\, J_{-\left[1+\frac{\beta}{\alpha}\right]}(x) \right] \qquad (1\text{-}6)$$

implying

$$v = \left(\frac{h_o}{h}\right)^{\left(1+\frac{\beta}{2\alpha}\right)} \cdot \left(\frac{g}{h_o}\right)^{\frac{1}{2}} \cdot \left[A J_{\beta/\alpha}(x) - B J_{-\beta/\alpha}(x) \right] \qquad (1\text{-}7)$$

where the J's are Bessel functions of the first kind and of order $\frac{\beta}{\alpha}$ or $1+\frac{\beta}{\alpha}$ and where $x = \frac{2\sigma}{\alpha (gh)^{\frac{1}{2}}}$ denotes the argument.

As long as $\frac{\beta}{\alpha}$ is not any integer, these functions ~~series~~ can be evaluated with the help of the series expansion

$$J_r(x) = \left(\frac{x}{2}\right)^{r} \sum_{j=0}^{\infty} \frac{(-)^j}{j!\,\Gamma(r+j+1)} \cdot \left(\frac{x}{2}\right)^{2j}$$

where Γ is the Gamma function.

For the special case $\beta = 0$ and $\alpha \neq 0$, the above solutions are in terms of Bessel functions of order 1 and 0, of the first and second kind, with a plus sign affixed in front of B in the formula for v.

For the case $\beta \neq 0$, $\alpha = 0$, the equation for Z becomes a homogeneous differential equation with constant coefficients which has for solution

$$Z = e^{-\frac{\beta y}{2}} \left[A e^{\lambda y/2} + B e^{-\lambda y/2} \right] \qquad (1\text{–}8)$$

implying

$$v = \frac{g}{2\sigma} \cdot \left[(\beta - \lambda) A e^{\lambda y/2} - (\beta + \lambda) B e^{-\lambda y/2} \right] e^{-\beta y/2} \qquad (1\text{–}9)$$

where

$$\lambda = i \left(\frac{4\sigma^2}{g h_o} - \beta^2 \right)^{\frac{1}{2}}$$

For instance Channel 3 in the 6 Channel Model could be schematized by a channel of constant depth and exponential width.

APPENDIX 2

Part 1

SOLUTIONS FOR THE ONE DIMENSIONAL MODELS

1 Channel

$$Z = \frac{\cos K(y-l)}{\cos Kl} \qquad v = \left(\frac{g}{1120}\right)^{\frac{1}{2}} \frac{\sin K(y-l)}{\cos Kl}$$

where	M_2	K_1
$Kl =$	180.7°	93.7°

2 Channels

$$Z_1 = \left(\frac{310}{h_1}\right)^{.7763} \left(A_1 J_{1.5526}(x_1) + B_1 J_{-1.5526}(x_1)\right)$$

$$Z_2 = B_2 \cos K_2 (y-1_2)$$

$$v_1 = \left(\frac{310}{h_1}\right)^{1.2763} \left(\frac{g}{310}\right)^{1/2} \left(A_1 J_{.5526}(x_1) - B_1 J_{-.5526}(x_1)\right)$$

$$v_2 = B_2 \left(\frac{g}{310}\right)^{1/2} \sin K_2 (y-1_2)$$

where

	M2	K1	UNITS
A_1 =	.63	-9.29	CM
B_1 =	-4.35	-2.18	CM
B_2 =	- .89	-3.15	CM
X_1 =	.9456 → 2.7349	.4902 → 1.4177	
$K_2 1_2$ =	110.2°	59.7°	

3 Channels

$$Z_1 = \left(\frac{310}{h_1}\right)^{.6968} \left(A_1 J_{1.3936}(x_1) + B_1 J_{-1.3936}(x_1)\right)$$

$$Z_2 = \left(\frac{310}{h_2}\right)^{-.1778} \left(A_2 J_{-.3555}(x_2) + B_2 J_{.3555}(x_2)\right)$$

$$Z_3 = B_3 \cos K_3 (y - l_3)$$

$$v_1 = \left(\frac{310}{h_1}\right)^{1.1968} \left(\frac{g}{310}\right)^{\frac{1}{2}} \left(A_1 J_{.3936}(x_1) - B_1 J_{-.3936}(x_1)\right)$$

$$v_2 = \left(\frac{310}{h_2}\right)^{.3223} \left(\frac{g}{310}\right)^{\frac{1}{2}} \left(A_2 J_{-1.3555}(x_2) - B_2 J_{1.3555}(x_2)\right)$$

$$v_3 = \left(\frac{g}{800}\right)^{\frac{1}{2}} B_3 \sin K_3 (y - l_3)$$

where

	M2	K1	UNITS
A_1 =	.25	-6.98	CM
B_1 =	-3.46	-1.99	CM
A_2 =	-3.11	-6.21	CM
B_2 =	-2.90	1.01	CM
B_3 =	-1.26	-2.79	CM
X_1 =	.8281 → 2.3939	.4293 → 1.2409	
X_2 =	2.8735 → 2.5330	1.4896 → 1.3133	
$K_3 l_3$ =	110.2°	59.7°	

6 Channels

$$Z_1 = \left(\frac{1600}{h_1}\right)^{.6935} \left(A_1 J_{1.3969}(x_1) + B_1 J_{-1.3969}(x)\right)$$

$$Z_2 = \left(\frac{310}{h_2}\right)^{.6967} \left(A_2 J_{1.3934}(x_2) + B_2 J_{-1.3934}(x_2)\right)$$

$$Z_3 = \left(\frac{310}{h_3}\right)^{-.1778} \left(A_3 J_{-.3555}(x_3) + B_3 J_{.3555}(x_3)\right)$$

$$Z_4 = \left(\frac{400}{h_4}\right)^{.7950} \left(A_4 J_{1.5918}(x_4) + B_4 J_{-1.5918}(x_4)\right)$$

$$Z_5 = A_5 \cos K_5 y + B_5 \sin K_5 y$$

$$Z_6 = \left(\frac{250}{h_6}\right)^{.6994} \left(A_6 J_{1.3987}(x_6) + B_6 J_{-1.3987}(x_6)\right)$$

$$v_1 = \left(\frac{1600}{h_1}\right)^{1.1965} \left(\frac{g}{1600}\right)^{\frac{1}{2}} \left(A_1 J_{.3969}(x_1) - B_1 J_{-.3969}(x_1)\right)$$

$$v_2 = \left(\frac{310}{h_2}\right)^{1.1967} \left(\frac{g}{310}\right)^{\frac{1}{2}} \left(A_2 J_{.3934}(x_2) - B_2 J_{-.3934}(x_2)\right)$$

$$v_3 = \left(\frac{310}{h_3}\right)^{.3228} \left(\frac{g}{310}\right)^{\frac{1}{2}} \left(A_3 J_{-1.3555}(x_3) - B_3 J_{1.3555}(x_3)\right)$$

$$v_4 = \left(\frac{400}{h}\right)^{1.2959} \left(\frac{g}{400}\right)^{\frac{1}{2}} \left(A_4 J_{.5918}(x) - B_4 J_{-.5918}(x_4)\right)$$

$$v_5 = \left(\frac{g}{800}\right)^{\frac{1}{2}} \left(A_5 \sin K_5 y - B_5 \cos K_5 y\right)$$

$$v_6 = \left(\frac{250}{h_6}\right)^{1.1994} \left(\frac{g}{250}\right)^{\frac{1}{2}} \left(A_6 J_{.3987}(x_6) - B_6 J_{-.3987}(x_6)\right)$$

Where

	WITHOUT OUTFLOW		WITH OUTFLOW		
	M2	K1	M2	K1	UNITS
A_1 =	2.68	- 2.93	3.04	-2.11	CM
B_1 =	.06	- 2.45	.52	-2.17	CM
A_2 =	-3.03	- 10.90	-2.45	-8.07	CM
B_2 =	-2.40	- .63	-2.56	- .69	CM
A_3 =	-2.69	- 10.50	-3.45	-7.83	CM
B_3 =	-2.04	2.42	- .93	2.40	CM
A_4 =	2.87	16.80	2.32	11.80	CM
B_4 =	.81	2.92	- .29	1.84	CM
A_5 =	- .15	- 3.36	.42	-2.07	CM
B_5 =	- .82	- 2.94	- .83	-2.17	CM
A_6 =	1.30	3.29	2.03	.93	CM
B_6 =	1.03	1.41	.43	.63	CM
OUTFLOW	0	0	2.50	1.21	$10^5 M^3/SEC$

ARGUMENTS

x_1 =	2.1779	2.7750	1.1285	1.4395
x_2 =	.5481	1.2435	.2839	.6440
x_3 =	2.8735	2.5331	1.4900	1.3131
x_4 =	1.4704	.9884	.7622	.5123
$K_5 l_5$ =	69.2°		35.9°	
x_6 =	.3582	.6741	.1857	.3495

APPENDIX 2

Part 2

CURRENTS FOR THE 6 CHANNEL MODEL

Junction	WITHOUT TRANSPORT M 2	K 1	WITH TRANSPORT M 2	K 1	UNITS
0	-3.3	.6	-9.7	.3	
1	-1.8	1.7	-3.8	1.1	
2	1.6	17.0	10.0	12.0	
3	21.0	18.0	16.0	13.0	
4	6.8	4.9	7.0	3.6	
5	1.8	.6	5.2	.8	
6	0	0	24.0	2.3	CM/SEC
TRANSPORT AT THE HEAD	0	0	17.5	1.7	$10^6 M^3$/SEC

These values are obtained assuming the boundary value Z_o at the mouth to be 70 cm and 14 cm for M_2 and K_1 respectively.

0 denotes the mouth of the sea; 1, the junction of channels 1 and 2, etc.

APPENDIX 2

Part 3

COMPARISON BETWEEN THE TRANSPORT bhv FOR THE 6 CHANNEL MODEL AND THAT FOR THE THREE BASIN MODEL

y	6 Channel	3 Basin
0	-24.4	$-9.25 \longrightarrow -11.9$
6	-4.86	$-2.79 \longrightarrow -5.12$
10	1.30	$2.95 \longrightarrow \infty$
11.4	1.92	
12		$3.93 \longrightarrow \infty$
14	2.97	$4.21 \longrightarrow 4.39$
22	2.21	$2.93 \longrightarrow 3.35$
23.5×10^2 km	1.74	$2.03 \longrightarrow 2.50$ $\times 10^6$ m^3/sec

APPENDIX 3

CONSTANTS

The upper number refers to M_2 ; the lower one to K_1.

$$\sigma = \begin{pmatrix} 1.4059 \\ .7292 \end{pmatrix} \times 10^{-4} \text{ sec}^{-1} \qquad g = 981 \text{ cm/sec}^2$$

$$2a_1 = 7.62\times10^7 \text{ cm} \qquad 2a_2 = \begin{pmatrix} 3.54 \\ 1.77 \end{pmatrix}\times10^7 \text{ cm} \qquad 2a_3 = 4.39\times10^7 \text{ cm}$$

$$h_1 = 1.57\times10^5 \text{ cm} \qquad h_2 = 3.7\times10^5 \text{ cm} \qquad h_3 = 8.0\times10^5 \text{ cm}$$

$$2L_1 = 10\times10^7 \text{ cm} \qquad 2L_2 = 2\times10^7 \text{ cm} \qquad 2L_3 = \begin{pmatrix} 14.5 \\ 11.5 \end{pmatrix}\times10^7 \text{ cm}$$

$$2\omega_1 = 1.2913\times10^{-4} \text{ sec}^{-1} \qquad 2\omega_2 = 1.3486\times10^{-4} \text{ sec}^{-1} \qquad 2\omega_3 = 1.3955\times10^{-4} \text{ sec}^{-1}$$

$$\lambda_1 = .2525 \qquad \lambda_2 = \begin{pmatrix} .2527 \\ .1264 \end{pmatrix} \qquad \lambda_3 = .2200$$

$$\lambda_1' = 1.0411\times10^{-8} \text{cm}^{-1} \qquad \lambda_2' = 2.2424\times10^{-8} \text{cm}^{-1} \qquad \lambda_3' = 1.5743\times10^{-8} \text{cm}^{-1}$$

$$\lambda_1' a_1 = .3967 \qquad \lambda_2' a_2 = \begin{pmatrix} .3969 \\ .1985 \end{pmatrix} \qquad \lambda_3' a_3 = .3456$$

$$K_o^{(1)\prime} = \begin{pmatrix} 1.1335 \\ .5875 \end{pmatrix} \times10^{-8} \text{cm}^{-1} \qquad K_o^{(2)\prime} = \begin{pmatrix} 2.3377 \\ 1.2118 \end{pmatrix} \times10^{-8} \text{cm}^{-1} \qquad K_o^{(3)\prime} = \begin{pmatrix} 1.5861 \\ .8222 \end{pmatrix} \times10^{-8} \text{cm}$$

$$2K_o^{①} l_1 = \begin{pmatrix} 1.1335 \\ .5876 \end{pmatrix} \text{ rad} \qquad 2K_o^{②} l_2 = \begin{pmatrix} .4675 \\ .2424 \end{pmatrix} \text{ rad} \qquad 2K_o^{③} l_3 = \begin{pmatrix} 2.2998 \\ .9455 \end{pmatrix} \text{ rad}$$

$$\alpha_1 = 2.4725\text{x}10^4 \text{ sec}^2 \qquad \alpha_2 = \begin{pmatrix} 1.1486 \\ .5743 \end{pmatrix} \text{x}10^4 \text{ sec}^2 \qquad \alpha_3 = 1.4244\text{x}10^4 \text{sec}^2$$

$$2\omega_1 \alpha_1 = 3.1927 \text{ sec} \qquad 2\omega_2 \alpha_2 = \begin{pmatrix} 1.5490 \\ .7745 \end{pmatrix} \text{sec} \qquad 2\omega_3 \alpha_3 = 1.9878 \text{ sec}$$

$$\sigma \alpha_1 = \begin{pmatrix} 3.4761 \\ 1.8029 \end{pmatrix} \text{sec} \qquad \sigma \alpha_2 = \begin{pmatrix} 1.6148 \\ .4188 \end{pmatrix} \text{ sec} \qquad \sigma \alpha_3 = \begin{pmatrix} 2.0026 \\ 1.0386 \end{pmatrix} \text{ sec}$$

$$(h_1/g)^{\frac{1}{2}} = 12.6433 \text{ sec} \qquad (h_2/g)^{\frac{1}{2}} = 6.1301 \text{ sec} \qquad (h_3/g)^{\frac{1}{2}} = 9.0355 \text{ sec}$$

$$F^{①} = .9746$$

$$G^{①} = \begin{pmatrix} .1095 \\ .0537 \end{pmatrix} \qquad G^{③} = \begin{pmatrix} .3753 \\ .1841 \end{pmatrix}$$

Note: All the calculations have been carried through in cgs although many values of length or volume are quoted in other units in the body of the thesis for easier apprehension.

Manuscript Report Series No. 3

THE ANALYSIS OF NINETEEN YEARS OF OBSERVATIONS ON THE HIGH AND LOW WATER WITH THE AID OF THE GERMAN METHOD

Gabriel Godin, S. E. Eldring and J. D. Taylor

1967

ROGER DUHAMEL, F.R.S.C.
QUEEN'S PRINTER AND CONTROLLER OF STATIONERY
OTTAWA, 1967

CONTENTS

LIST OF FIGURES AND TABLES

Sommaire

Les principes de la méthode allemande pour l'analyse des marées de petites profondeurs sont d'abord exposés dans la section 1. I. Les instructions au programmeur pour mettre en pratique cette méthode sur un ordinateur electronique sont données dans la section 1. II. Des tables contenant les résultats d'analyses à Québec et à Grondines, des comparaisons entre les valeurs prédites et observées à ces deux ports durant 1965, et les matrices nécessaires à l'analyse concluent cette partie du rapport. Le travail préliminaire pour la transformation des données brutes en données utilisables est décrit dans la section 2. Enfin le contenu des programmes pour l'analyse de données continues et discontinues sur le CDC 3100 est esquissé dans la dernière section.

Summary

The principles of the German method for the analysis of shallow water tides are described in section 1. I. The instructions to the programmer necessary to put this method into practice on an electronic computer are given in section 1. II. Tables giving the results of analyses at Québec and Grondines, a comparison between the predicted and observed tides at these two ports during 1965 and the matrices necessary for the performance of the analysis conclude this part of the report. The preliminary work necessary to transform the raw data into a format suitable for analysis is discussed in section 2. Finally the content of the programs for continuous and discontinuous data is displayed in the last section.

ACKNOWLEDGEMENTS

The work was undertaken at the request of Mr. G. C. Dohler, Officer-in-Charge, Tides, Currents and Water Levels Section, Canadian Hydrographic Service. This section also undertook to abstract and prepare the data in a format suitable for analysis. The German Hydrographic Institute, in particular Messrs. W. Horn, Regierungsdirektor, W. Habich, and Dr. K. Munkelt have helped considerably by supplying a wealth of technical detail on the method they have used so successfully over many years. The Observatories Branch was responsible for the writing of the Lunar Transit Program. Mr. Maurice Isabelle provided the illustrations.

PRINCIPES DE LA MÉTHODE

THÉORIE ET RÉSULTATS

1.- La Correlation entre le Passage de la Lune et l'Apparition des Pleines et Basses Mers.

La lune et le soleil créent sur la terre un champ de gravitation qui se manifeste par le phénomène des marées. Si nous ne considérons que la lune pour le moment et si nous traçons sur chaque point de la terre les vecteurs de la force de gravitation exercée par cet astre à un moment donné, nous notons que les pointes de ces vecteurs forment une ellipsoide dont l'axe principal est dirigé vers la lune.

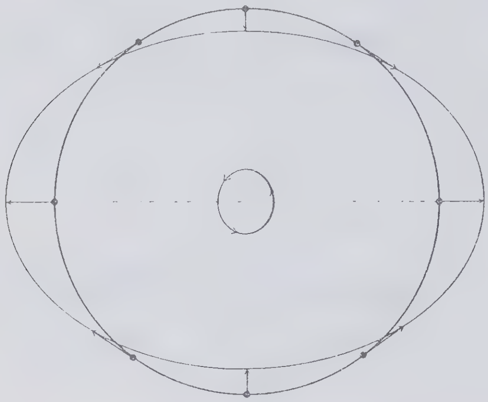


Fig. 1. Champ des forces de gravitation causées par la lune.

La lune bouge assez peu au cours d'une journée alors que la terre subit une rotation complète dans le même temps. Un point P sur la terre subira donc deux fois au cours d'une journée un maximum des forces de la marée; cette force sera dirigée dans la même direction, du moins dans le cadre de référence d l'observateur. Les particules au point P, à cause de leur inertie, n'obéissent pas immédiatement à cette force, mais en général, le déplacement dû à la marée passera par un maximum après le passage de la lune au zénith et au nadir de P.

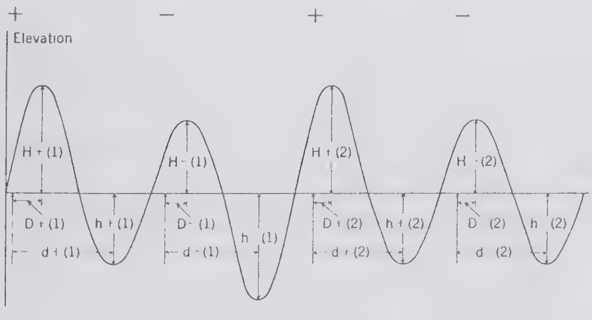
En fait la situation n'est pas aussi simple que nous l'avons présentée: le soleil ajoute son attraction à celle de la lune, la lune tourne autour de la terre, la terre tourne autour de soleil, le plan de la rotation de la lune n'est pas parallèle à l'écliptique, les courbes de révolution sont des ellipses et non des cercles, l'orbite de la lune n'est pas stable, l'axe de la terre oscille etc.,etc.

Ces phénomènes additionels introduisent des modulations diurnes, hebdomadaires, mensuelles, annuelles, seculaires, etc.,etc., dans les maxima et les minima observés, mais quand même on peut presque toujours faire une correlation exacte entre le passage de la lune, soit au zénith, soit au nadir, et l'apparition d'un maximum ou d'un minimum de la marée.

Cette correlation devient plus difficile lorsque la marée est de caractère mixte ou diurne; ce type de marée apparait dans certaines portions des bassins océaniques ou dans le voisinage des points d'amphidromie semidiurnes. Dans ce cas, si l'on suit un maximum au cours de jours lunaires successifs, ce maximum diminuera en amplitude et deviendra à peu près imperceptible à un jour donné. Par après il se remettra à grandir, mais il y a une bonne chance qu'on l'ait perdu en chemin.

Nous nous restreignons à la marée verticale dans le fleuve Saint-Laurent; là la marée est de caractère semidiurne. Nous pouvons donc y faire la correlation entre la passage de la lune et les maxima ou les minima sans aucune ambiguité; il nous suffit pour celà de suivre un calendrier lunaire. En effet la lune se meut un peu au cours d'un jour solaire et il s'ensuit qu'elle prend un peu plus que 24 heures solaires pour réapparaitre au même point dans le ciel (1 jour lunaire = 1.035050 jour solaire moyen). Nous montrons dans la figure 2 comment s'effectue cette correlation.

Hauteur
du niveau
l'eau



temps

Fig. 2 Correlation entre le passage de la lune au zénith (+) ou au nadir (-) et la hauteur (H ou h) et le retard (D ou d) des extrema du niveau d'eau.

$+$ et $-$ indiquent le passage de la lune au zénith ou au nadir et le temps de ce passage est noté par une flèche sur l'échelle du temps.

$H_{+-}(j)$ dénote la hauteur du maximum qui suit le passage de la lune au zénith ou au nadir, ; $D_{+-}(j)$ est le temps écoulé entre la passage de la lune et l'apparition du maximum. $h_{+-}(j)$ et $d_{+-}(j)$ sont les mêmes variables mais se rapportant au minimum.

Nous n'avons pas à commencer nos observations à la suite d'un passage au zénith: nous sommes libres de commencer par la première observation qui se présente, que ce soit un maximum ou un minimum, et qu'elle se rapporte à un passage au zénith ou au nadir. Mais il faut prendre soin de noter si les deux premières observations se rapportent au même passage ou non et si nous avons affaire d'abord à un transit au zénith ou au nadir; tout simplement afin de reconnaitre à quoi se rapportent les valeurs analysées. Il se peut qu'à un certain point de la terre, le maximum suive de fort loin le passage de la lune et que le minimum qui suive ne se rapporte pas au même passage.

Une fois que nous avons fixé l'ordre des pleines et basses mers par rapport au passage de la lune, nous pouvons former les huit suites d'observations

$$\{H_{\pm}(j)\}, \{D_{\pm}(j)\}, \{h_{\pm}(j)\}, \{d_{\pm}(j)\} \quad j = 0, \pm 1, \pm 2, \ldots, \pm N \qquad (1)$$

dans l'ordre qui nous plait.

Ces huit suites peuvent être considéréees comme une seule suite de vecteurs z(j) à huit dimensions:

$$\{z(j)\} \qquad (2)$$

ce qui simplifie considérablement la notation.

Dans notre cas nous choisissons l'arrangement

$$z(j) = (d_+(j),\ h_+(j), D_+(j), H_+(j), d_-(j), h_-(j), H_-(j), D_-(j) \qquad (3)$$

2.- Le Repliement des Fréquences

La variable indépendante dans la suite (2) est j, le nombre de jours lunaires écoulés depuis une origine donnée.

z(j) résulte dans de l'échantillonnage de la courbe des observations à intervalles d'un jour lunaire; une telle procédure <u>viole la loi d'échantillonnage</u> qui dit que le pas de temps Δt entre deux données doit satisfaire

$$2 \Delta t \, \sigma_m \leqslant 1$$

σ_m étant la plus grande fréquence contenue dans les oscillations. Dans le cas des marées, σ_m est de l'ordre de 1/2 cycle/heure, ce qui nécessite $\Delta t = 1$ heure. Si nous excédons ce pas de temps, nous causons le <u>phénomène du repliement des fréquences</u>: une fréquence σ qui dépasse la <u>fréquence de repliement</u> $\sigma_r = 1/2\Delta t$ se manifeste par une fréquence <u>inférieure</u> à σ_r et la grandeur de cette fréquence repliée dépend de la distance entre σ et σ_r, sur l'axe des fréquences. On peut considérer la fréquence σ_r comme un point autour duquel s'effectue le repliement des fréquences supérieures. La figure 3 illustre ce phénomène d'une façon schématique. Dans notre cas $\sigma_r = 1/2$ cycle/jour lunaire.

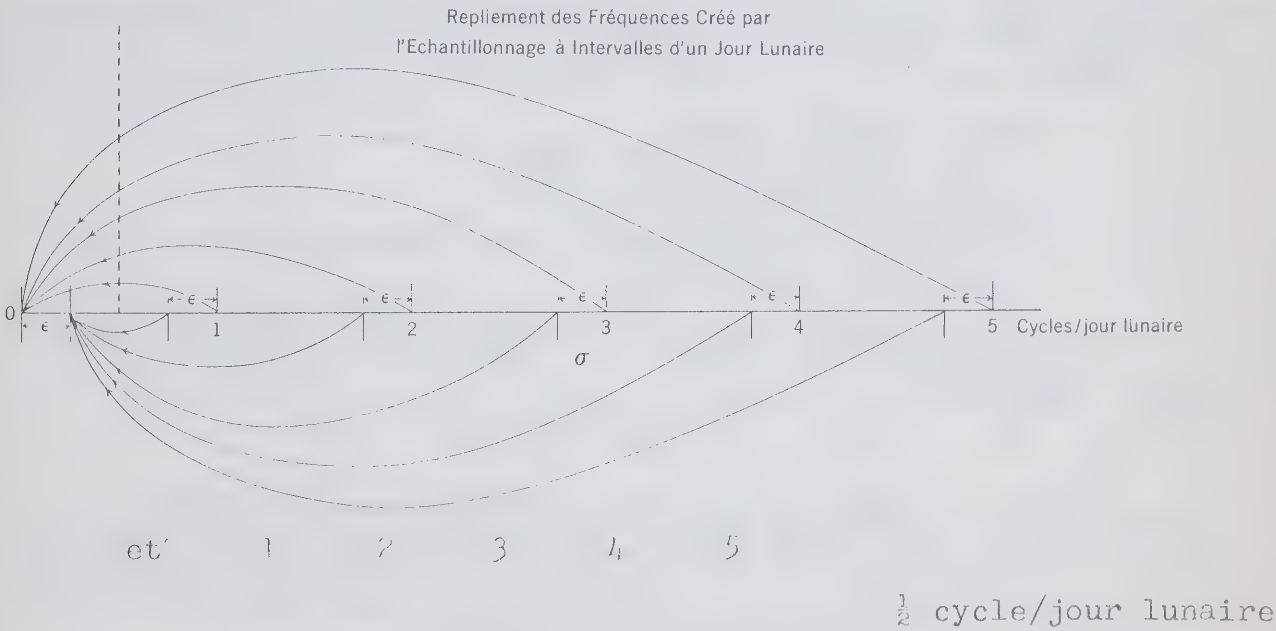


Fig. 3 Repliement des fréquences supérieures à ½ cycle/jour lunaire causé par l'échantillonnage d'une fonction à intervalles d'un jour lunaire.

Loin d'être désastreux, ce repliement est d'un grand avantage pratique dans le cas des marées de petites profondeurs. En effet, à cause des effets non linéaires dans les bassins de petites profondeurs, ces marées contiennent une masse de fréquences dont nous ne pouvons prévoir ni la valeur, ni l'importance, ce qui rend fort difficile une analyse directe de ce genre d'observations. L'échantillonnage à intervalles d'un jour lunaire,

à cause du repliement, réduit considérablement le nombre des fréquences contenues dans $\{z(j)\}$. Si nous appelons <u>fréquence effective</u>, la fréquence repliée, et <u>fréquence vraie</u>, la fréquence originale, nous voyons que nous avons affaire seulement aux fréquences effectives dans la suite $\{z(j)\}$.

Pour connaitre la fréquence effective dérivée par repliement d'une fréquence vraie, nous pouvons nous servir des <u>nombres de Doodson</u> qui caractérisent chacune des ondes composantes de la marée. Les nombres de Doodson consistent en une suite de six nombres entiers que nous pouvons écrire sous la forme

$$(i_o, j_o, k_o, l_o, m_o, n_o)$$

et qui déterminemt automatiquement la phase et la fréquence d'une onde donnée. La fréquence vraie σ est

$$\sigma = i_o \dot{\tau} + j_o \dot{h} + k_o \dot{s} + l_o \dot{p} + m_o \dot{N}' + n_o \dot{p}'$$

le point dénote le taux de changement de la variable par unité de temps,

τ = le temps lunaire moyen
s = la longitude moyenne de la lune
h = la longitude moyenne du soleil
p = la longitude du périgée
N' = -N, N étant la longitude du noeud ascendant de la lune
p' = la longitude du périhélie

Le changement de phase d'une onde composante durant un jour lunaire est

$$\Delta \varphi = i_o + j_o \Delta h + k_o \Delta s + l_o \Delta p + m_o \Delta N' + n_o \Delta p' \qquad (4)$$

puisque $\Delta \tau = 1$ dans ce cas. Ce changement de phase est identique à celui d'une autre onde composante caractérisée par les nombres $(i_o', j_o, k_o, l_o, m_o, n_o)$ puisque i_o et i_o' sont des nombres entiers et qu'ils équivalemt à des multiples de $360°$ dans le mesure habituelle des angles.

Nous avons donc la règle suivante: l'onde vraie $(i_o, j_o, k_o, l_o, m_o, n_o)$ devient l'onde effective $(j_o, k_o, l_o, m_o, n_o)$ par l'échantillonnage à intervalles d'un jour lunaire. Ce qui équivaut à dire que nous éliminons la variable τ par une telle procédure.

Par exemple, les ondes composantes $M_1, M_2, M_3, M_4, \ldots$ qui sont caractérisées par les nombres de Doodson

$$(j,0,0,0,0,0) \qquad j=1,2,3,4,\ldots.$$

deviennent par repliement l'onde effective

$$(0,0,0,0,0)$$

ou tout simplement une constante d'addition, alors que

$MSM(0,-2,1,0,0)$, $\nu_2(2,-1,2,-1,0,0)$, $\lambda_2(2,1,-2,1,0,0)$,

$SN_4(4,1,-2,1,0,0)$, $MSN_6(6,1,-2,1,0,0)$, $2MSN_8(8,1,-2,1,0,0)$

donnent l'onde effective

$$(1,-2,1,0,0)$$

L'inverse toutefois n'est pas possible: nous ne pouvons pas déduire la fréquence vraie à partir de la fréquence effective. Mais ceci n'a pas d'intérêt immédiat puisque $z(j)$ est déterminé exclusivement par les fréquences effectives et nous pouvons l'écrire sous la forme

$$z(j)=\sum_{k=-n}^{n} a_k e^{2\pi i \sigma_k j} \qquad (5)$$

σ_k = la fréquence effective
a_k = un vecteur à huit dimensions donnant l'amplitude complexe de l'onde effective k
n = le nombre des ondes composantes effectives (assurément plus petit que celui des ondes vraies)

L'appendice 1 donne la liste des fréquence effectives suggérées par l'institut hydrographique allemand. Le nombre n_o est ignoré et $n_o p'$ est considéré comme une constante additive de phase.

Cependant nous ne pouvons pas suivre les variations nodales des ondes individuelles que contribuent à une onde effective donnée: l'emploi des ondes effectives nous oblige donc à analyser une suite d'observations qui a une durée d'au moins dix neuf ans afin d'éliminer cette modulation inconnue.

3.- Le Lissage des Données et les Tendances à Long Terme

Nous lissons les données à l'aide de l'opérateur

$$\mathcal{L} = \mathcal{A}_2^2 \, \mathcal{A}_3$$

et nous filtrons par après les données lissées à l'aide du filtre passes-bas

$$\mathcal{F} = \mathcal{A}_{26} \, \mathcal{A}_{28} \, \mathcal{A}_{31}$$

L'opérateur $\mathcal{A}_k$ indique le calcul de la moyenne arithmétique de k observations consécutives. Par exemple

$$\mathcal{A}_k z \equiv \frac{1}{k} \sum_{j=1}^{k} z(1+j) \equiv \bar{z} \quad (1)$$

Un produit symbolique d'opérateurs est équivalent à des convolutions répétées sur la même suite, dans le temps. (Godin 1966)

Le spectre de $\mathcal{A}_k$ est

$$A_k(\sigma) = \frac{\sin k\pi\sigma}{k\sin\pi\sigma}$$

Il s'ensuit que les spectres de $\mathcal{L}$ et $\mathcal{F}$ sont

$$L(\sigma) = \left(\frac{\sin 2\pi\sigma}{2\sin\pi\sigma}\right)^2 \left(\frac{\sin 3\pi\sigma}{3\sin\pi\sigma}\right)$$

$$F(\sigma) = \left(\frac{\sin 26\pi\sigma}{26\sin\pi\sigma}\right) \left(\frac{\sin 28\pi\sigma}{28\sin\pi\sigma}\right) \left(\frac{\sin 31\pi\sigma}{31\sin\pi\sigma}\right)$$

L'opérateur $\mathcal{L}$ déforme un peu les fréquences repliées et l'on doit corriger cette déformation à la fin de l'analyse: l'appendice 3 donne cette correction. Le filtre $\mathcal{F}$ ne cause pas de déformation appréciable. La Fig. 4 montre l'effet du lissage sur un échantillon des données.

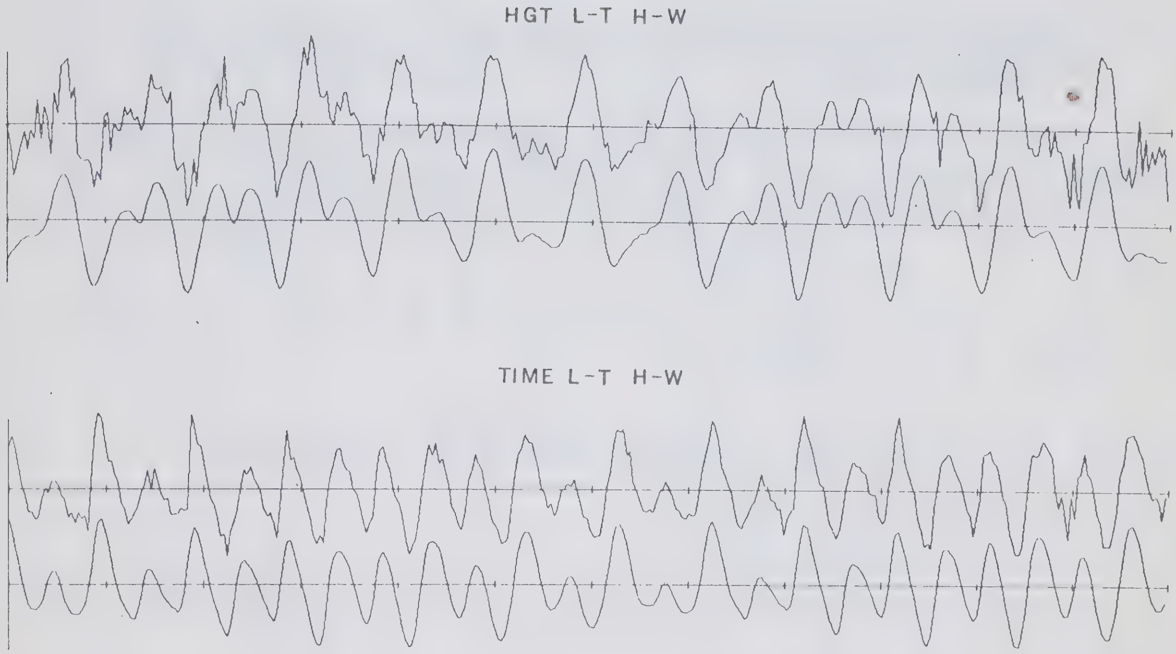


Fig. 4. Effet de l'opérateur lisseur sur une suite de données.

Le filtrat $z_o(j)$ doit être enlevé des données afin d'éliminer l'effet des tendances à long termes et l'analyse porte donc sur les résidus

$$z_r(j)=z(j)-z_o(j)$$

Dans ce qui suit nous n'écrivons pas le r afin d'alléger la notation mais il est toujours impliqué.

L'on se sert du filtrat pour la préparation des prédictions; on doit alors l'extrapoler.

4.- L'Analyse des Données à l'Aide de la Condition des Moindres Carrés.

Après un lissage préalable des données et un filtrage qui enlève les tendances à long terme, les inconnues a_k sont calculées en imposant la restriction (Horn 1960)

$$\sum_{j=-N}^{N} \left| z(j)-\sum_{k=-n}^{n} a_k e^{2\pi i \sigma_k j} \right|^2 = \text{Min} \qquad (6)$$

2N+1 est le nombre des données; 6689 dans notre cas.

L'équivalent de (6) est

$$\sum_{j=-N}^{N} \frac{\partial}{\partial a_\ell} \left| z(j) - \sum_{k=-n}^{n} a_k e^{2\pi i \sigma_k j} \right|^2 = 0 \text{ pour}$$

$$\ell = 0, \pm 1, \pm 2, \ldots, \pm n \qquad (7)$$

Or nous avons

$$\frac{\partial}{\partial a_\ell}\left(z(j)-\sum_{k=-n}^{n} a_k^* e^{-2\pi i \sigma_k j}\right)\left(z(j)-\sum_{k=-n}^{n} a_k e^{2\pi i \sigma_k j}\right)$$

$$=-\left[z(j)-\sum_{k=-n}^{n} a_k^* e^{-2\pi i \sigma_k j}\right] e^{2\pi i \sigma_\ell j}$$

Nous pouvons donc écrire (7) sous la forme

$$\sum_{j=-N}^{N} z(j)e^{2\pi i\sigma_\ell j} \equiv Z_\ell = \sum_{k=-n}^{n} a_k^* \sum_{j=-N}^{N} e^{2\pi ij(\sigma_\ell-\sigma_k)}$$

$$\equiv \sum_{k=-n}^{n} a_k^* \mathcal{a}_{k\ell}$$

ou plus simplement

$$Z_\ell = \sum_{k=-n}^{n} a_k^* \mathcal{a}_{k\ell} \qquad \ell=0,\pm1,\ldots,\pm n \tag{8}$$

où nous avons défini

$$\mathcal{a}_{k\ell} \equiv \sum_{j=-N}^{N} e^{2\pi ij(\sigma_\ell-\sigma_k)} = \frac{\sin\,(2N+1)\pi\,(\sigma_k-\sigma_\ell)}{\sin\pi\,(\sigma_k-\sigma_\ell)}$$

les fréquences étant mesurées en cycles/jour lunaire.

Le système d'équations linéaires (8) peut être écrit sous forme d'une équation de matrices:

$$Z = a^*\mathcal{a} \tag{9}$$

Z et a^* étant les vecteurs

$$Z=(Z_{-n},\ldots,Z_o,\ldots,Z_n)$$

$$a^*=(a^*_{-n},\ldots,a^*_o,\ldots,a^*_n)$$

et $\mathcal{A}$ est une matrice de dimensions 2nx2n dont les éléments sont les $\mathcal{A}_{k\ell}$.

a_k^* est la conjuguée complexe de a_k et puisque les observations sont des quantités réelles, il s'ensuit que

$$a_{-k}^* = a_k$$

Nous utilisons maintenant les propriétés de symétrie de la matrice $\mathcal{A}$ afin de réduire le système (9) à deux systèmes indépendants d'équations. Cette matrice peut être structurée de la façon suivante:

$$\begin{array}{c|c|c}
 & \ell = 0\ 1\ 2\ \ldots\ n & 0\ -1\ -2\ \ldots\ -n \\
\hline
k = 0, 1, 2, \ldots, n & \mathcal{A}_+ & \mathcal{A}_- \\
\hline
0, -1, -2, \ldots, -n & \mathcal{A}_- & \mathcal{A}_+
\end{array}$$

où nous avons défini

$$\mathcal{A}_+ = (\mathcal{A}_{k\ell}) = (\mathcal{A}_{-k-\ell})$$

$$\mathcal{A}_- = (\mathcal{A}_{k-\ell}) = (\mathcal{A}_{-k-\ell})$$

k et ℓ étant des nombres positifs dorénavant.

Nous réécrivons (9) sous la forme suivante:

$$\begin{aligned}
Z_+ &= a_+^* \mathcal{A}_+ + a_-^* \mathcal{A}_- \\
Z_- &= a_+ \mathcal{A}_+ + a_-^* \mathcal{A}_-
\end{aligned} \qquad (10)$$

où nous avons pris

$$Z=(Z_+ ,Z_-) \qquad a^* =(a^*_+ , a^*_-)$$

$$Z_+ =(Z_o,Z_1,\ldots,Z_n) \quad Z_-=(Z_o,Z_{-1},\ldots Z_{-n})$$

$$a^*_+=a_-=(a_o,a_{-1},\ldots,a_{-n}) \quad a^*_-=a_+ =(a_o,a_1,\ldots,a_n)$$

Deux combinaisons linéaires simples sur (10) donnent

$$(Z_+ + Z_-)=2Z_R \equiv 2c=(a_+ + a_-)(\mathcal{A}_+ + \mathcal{A}_-)=2a_R(\mathcal{A}_+ + \mathcal{A}_-) \equiv 2a_RC$$

$$(Z_+-Z_-)=2iZ_I \equiv 2is=(-a_+ + a_-)(\mathcal{A}_+ -\mathcal{A}_-)=-2ia_I(\mathcal{A}_+ -\mathcal{A}_-)\equiv 2ia_IS$$

qui a pour solution

$$a_R=cC^{-1}$$
$$-a_I=sS^{-1} \tag{11}$$

R et I indiquent la partie réelle ou imaginaire du nombre complexe. C et S sont les matrices $(\mathcal{A}_+ \pm \mathcal{A}_-)$. Il y a avantage à évaluer $-a_I$ plutôt que a_I puisque

$$z(j)=2\sum_{k=0}^{n} (a_R(k)\cos 2\pi\sigma_k j-a_I(k)\sin 2\pi\sigma_k j) \tag{12}$$

si nous écrivons (5) sous sa forme réelle qui est la forme habituelle.

Nous notons aussi que c_ℓ et s_ℓ signifient

$$c_\ell = \sum_{j=-N}^{N} z(j)\cos 2\pi\sigma_\ell j$$

$$s_\ell = \sum_{j=-N}^{N} z(j)\sin 2\pi\sigma_\ell j$$

5.- Les Calculs Numériques.

Nous voyons que l'analyse consiste à calculer les quantités c, s, C^{-1} et S^{-1}. Le calcul de C^{-1} et de S^{-1} peut être considéré comme un stage préliminaire à l'analyse puisqu'il peut se faire une fois pour toutes, ces matrices ne dépendant pas des observations $z(j)$, mais seulement des fréquences effectives σ_k et du nombre des données $2N+1$. Les vecteurs c et s doivent être évalués pour chaque analyse puisqu'ils sont déterminés par les observations.

Nous décrivons d'abord la méthode que nous avons suivie pour évaluer l'équivalent des matrices C^{-1} et S^{-1}, qui est celle de Banachiewicz-Cholesky-Crout et qui nous a été suggérée par le Dr. K. Munkelt de l'institut hydrographique allemand (Korn et Korn, 1961) Cette méthode permet le calcul rapide et efficace des vecteurs a_R et a_I à partir des vecteurs c et s; nous nous restreignons à la matrice C, la procédure à suivre étant identique pour la matrice S.

La matrice C est symétrique et par conséquent elle peut être representée par l'expression

$$C=(I-b^T)p(I-b) \tag{13}$$

I = la matrice unité dont les éléments sont donnés par

$$I_{k\ell} = \delta_{k\ell} = \begin{cases} 1 & k=\ell \\ 0 & k\neq\ell \end{cases}$$

p = une matrice diagonale dont les éléments doivent satisfaire

$$d_{k\ell} = 0 \quad k\neq\ell$$

b = une matrice triangulaire dont les éléments satisfont

$$b_{k\ell} = 0 \quad k\leq\ell$$

b^T= la transposée de b de telle sorte que $b^T_{\ell k} = b_{k\ell}$

Nous introduisons aussi la matrice

$$G=p(I-b) \tag{14}$$

Nous notons que

$$G_{kk} = \sum_{\ell=0}^{n} p_{k\ell}\,(\delta_{\ell k} - b_{\ell k}) = p_{kk}(1-b_{kk})$$

et que par conséquent

$$\boxed{p_{kk}=G_{kk}} \tag{15}$$

Puisque $b_{kk}=0$. La matrice diagonale p consiste donc des éléments diagonaux de G.

Nous notons aussi que

$$\boxed{G=C + b^T G} \tag{16}$$

$$\boxed{b = I - p^{-1}G} \tag{17}$$

Ces deux équations peuvent être comparées à des relations de recurrence: elles permettent l'évaluation progressive de G et de b, rangée par rangée, en alternant de l'une à l'autre, Voyons ceci:

Un élément de (16) ou de (17) est

$$G_{k\ell} = C_{k\ell} + \sum_{m=0}^{n} b^T_{km} G_{m\ell} = C_{k\ell} + \sum_{m=0}^{n} b_{mk} G_{m\ell} \tag{18}$$

$$b_{k\ell} = \delta_{k\ell} - \sum_{m=0}^{n} p^{-1}_{km} G_{m\ell} = \delta_{k\ell} - p^{-1}_{kk} G_{k\ell} = \delta_{k\ell} - G_{k\ell}/G_{kk}$$

$$= \begin{cases} 0 & k = \ell \\ -G_{k\ell}/G_{kk} & k \neq \ell \end{cases} \tag{19}$$

puisque p est diagonal et que $p^{-1}_{kk} = 1/p_{kk}$ alors que $p_{kk} = G_{kk}$.

Si maintenant nous considérons la nième rangée de G, nous avons

$$G_{n\ell} = C_{n\ell} + \sum_{k=0}^{n} b_{nk} G_{m\ell} = C_{n\ell}$$

puisque $b_{nk} = 0$ pour toutes les valeurs de k.

Donc les éléments de la nième rangée de G sont

$$\boxed{G_{n\ell} = C_{n\ell}} \quad \ell = 0,1,2,\ldots,n \tag{20}$$

De même

$$\boxed{b_{n\ell} = -G_{n\ell}/G_{nn}}, \quad \ell = 0,1,\ldots,n-1 \tag{21}$$

Nous obtenons la (n-1)ième rangée de G à l'aide de (18), puis celle de b à l'aide de (19) etc., jusqu'à ce que le calcul de G et b soit complété.

Nous sommes maintenant prets à évaluer l'équivalent de la matrice inverse C^{-1}. L'équation originale était

$$a_R C = c$$

ou

$$a_R(I-b^T)G=c$$

En multipliant par G^{-1} par la droite nous obtenons

$$a_R(I-b^T)= cG^{-1}$$

ou

$$\boxed{a_R = ab^T + cG^{-1} \equiv \alpha A} \tag{22}$$

une relation qui nous permet l'évaluation progressive des inconnues a_k à partir des quantités connues c_ℓ . Nous avons introduit deux nouvelles matrices α et A qui ont pour définition

$$\alpha_{k\ell} = \begin{cases} c_k & k \leq \ell \\ a_k & k > \ell \end{cases}$$

$$A_{\ell k} = \begin{cases} b^T_{\ell k} = b_{k\ell} & \ell < k \\ G^{-1}_{\ell k} & \ell \geqslant k \end{cases}$$

L'équation $a_R = \alpha A$ prend la forme suivante lorsque mise en diagramme:

$$a_R \quad = \quad \alpha \quad\quad A \tag{23}$$

$$(a_o, a_1, \ldots, a_n) = \left(\begin{pmatrix} c_o c_1 \ldots\ldots & c_n \\ a_o c_1 \ldots\ldots & c_n \\ & \\ a_o a_1 a_2 \ldots a_{n-1} & c_n \end{pmatrix} \begin{pmatrix} G^{-1}_{oo} & b_{1o} & \ldots & b_{no} \\ G^{-1}_{10} & G^{-1}_{11} & b_{21} \ldots & b_{n1} \\ & & & \\ G^{-1}_{no} & G^{-1}_{n1} \ldots & G^{-1}_{n,n-1} & G^{-1}_{nn} \end{pmatrix} \right)$$

Le calcul des a_n consiste à calculer le produit de c par la première colonne de A; ceci nous donne a_o que nous plaçons sur c_o dans l'ordinateur. Nous calculons alors les produit de $(a_o, c_1, \ldots, c_n)$ par la deuxième colonne de A; ce qui nous donne a_1. Nous continuons de la sorte jusqu'à toutes les inconnues soient calculées.

Nous appellons B la matrice qui correspond à S^{-1}; les matrices 2A et 2B sont données dans l'appendice 2. Nous avons ajouté le facteur 2 pour éviter l'introduction de ce même facteur lorsque nous voulons une expression de $z(j)$ sous forme réelle. Nous notons que ces matrices consistent de la juxtaposition des matrices G^{-1} et b. Il nous reste à indiquer comment l'on calcule G^{-1}.

Si nous multiplions (14) par p^{-1} par la gauche et par G^{-1} par la droite nous obtenons

$$G^{-1} = b^{-1} + bG^{-1} \tag{24}$$

dont un élément est

$$G^{-1}_{k} = \delta_{k\ell} / G_{k\ell} + \sum_{m=0}^{n} b_{km} G^{-1}_{m\ell} \tag{25}$$

Cette fois, nous procédons à partir de la rangée 0 qui est

$$G^{-1}_{0\ell} = \delta_{0\ell}/G_{0\ell}$$

Ensuite nous calculons la rangée 1 etc.etc.

Le calcul de c et de s constitue la partie laborieuse du calcul. Ces vecteurs ont pour expression

$$c_k = \sum_{j=-N}^{N} z(j)\cos 2\pi\sigma_k j \qquad (26)$$

$$s_k = \sum_{j=-N}^{N} z(j)\sin 2\pi\sigma_k j \qquad (27)$$

Afin d'effectuer ce calcul nous pouvons introduire une table de cosinus dans l'ordinateur; ceci prend de l'espace mais le calcul est rapide. Nous pouvons éviter l'emploi d'une table si nous nous servons d'une relation de recurrence entre les fonctions trigonométriques mais l'épargne d'espace est contrebalancée par un calcul plus lent. M.J. Taylor emploie un compromis entre ces deux techniques: ce compromis est exposé dans son rapport.

Avant de terminer ce paragraphe nous devons rappeler au lecteur que z(j) est un vecteur à huit dimensions dans notre cas et que les opérations (11) à (27) doivent porter sur chacune des composantes des vecteurs z(j), c_k ou s_k.

6.-L'Erreur probable.

Cette erreur est la même pour toutes les composantes des vecteurs a_R et a_I, et elle est égale à

$$\Delta = \sqrt{\frac{1}{2N^2}\sum_{j=-N}^{N}\left[z(j) - \sum_{k=-m}^{m} a_k e^{2\pi i\sigma_k j}\right]^2} \qquad (28)$$

Nous pouvons évaluer cette erreur d'une façon approximative mais beaucoup plus rapidement si nous appliquons un opérateur éliminateur de marée aux observations; un tel opérateur pourrait être (Godin 1966a)

$$\frac{1}{16} S_{22} S_{24} S_{26} S_{28} \tag{29}$$

S_j indique la soustraction de deux données séparées par j unités de temps. Le spectre et la représentation temporelle de (29) sont

$$\sin 22\pi\sigma \ \sin 24\pi\sigma \ \sin 26\pi\sigma \ \sin 28\pi\sigma \tag{30}$$

et

$$\{e_j\} \quad j = 0, \pm 1, \pm 2, \ldots \tag{31}$$

où

$$e_j = \begin{cases} 2 & j = 0 \\ 1 & j = \pm 2, \pm 4, \pm 5 \\ -1 & j = \pm 22, \pm 24, \pm 26, \pm 28 \\ 0 \text{ pour toutes les autres valeurs de } j & \end{cases}$$

L'emploi de l'éliminateur (29) nous donne l'approximation à l'erreur probable

$$\Delta \sim \sqrt{\frac{\sum_{j=-N_o}^{N_o} n^2(j)}{(2N_o + 1)}} \tag{32}$$

$n(j)$ représente le nombre obtenu par l'application de l'opérateur (29) à une suite de 99 nombres extraits de la suite $\{z(j)\}$. Par exemple

$$n(j) = (S_{24}S_{25})^2 \left\{ n(j) - \frac{1}{16} S_{22}S_{24}S_{26}S_{28} \right\}$$

$$= 2z(j) + z(j+2) + z(j-2) + z(j+4) + z(j-4) - z(j+22)$$

$$- z(j-22) - z(j+24) - z(j-24) - z(j+26) - z(j-26) - z(j+28)$$

$$- z(j-28) + z(j+50) + z(j-50)$$

$2N_o+1$ est le nombre de $n(j)$ que nous nous donnons la peine de calculer et il est certainement plus petit que $2N+1$.

7.-L'Analyse d'observations discontinues.

Nous pouvons ajuster la fonction

$$\sum_{k=-n}^{n} a_k e^{2\pi\sigma_k j}$$

à une suite de $2M+1$ segments d'observations, d'une durée individuelle de $2N_o$ jours lunaires, séparés par $2M$ brisures d'une durée de $Q-1$ jours lunaires en imposant la condition des moindres carrés

$$\sum_{\ell=-M}^{M} \sum_{j=-N_o}^{N_o} z(j) e^{2\pi i \sigma_\ell (j+\ell(Q+2N_o+1))} =$$

$$\sum_{k=-n}^{n} a_k \sum_{\ell=-M}^{M} \sum_{j=-N_o}^{N_o} e^{2\pi i(\sigma_\ell - \sigma_k)\left[j+\ell(Q+2N_o+1)\right]} \qquad (33)$$

La somme d'exponentielles du cote droit de l'expression se simplifie de la facon suivante:

$$\sum_{\ell=-M}^{M} e^{2\pi i(\sigma_\ell-\sigma_k)\left[\ell(Q+N_o+1)\right]} \quad \sum_{j=-N_o}^{N_o} e^{2\pi i(\sigma_\ell-\sigma_k)j}$$

$$\frac{\sin\left[(2M+1)\pi(\sigma_k-\sigma_\ell)(Q+2N_o+1)\right]}{\sin} \qquad \frac{\sin\left[(2N_o+1)\pi(\sigma_k-\sigma_\ell)\right]}{\sin\pi(\sigma_k-\sigma_\ell)}$$

$$\equiv a_{k\ell}(Q,M)$$

Nous notons que

$$a_{k\ell}(0,0)=\frac{\sin\quad(2N_o+1)\pi(\sigma_k-\sigma_\ell)}{\sin} = a_{k\ell}$$

tel que nous l'avions defini auparavant dans le paragraphe 3.

Les inconnues a_R et a_I s'obtiennent dans de cas en calculant

$$a_R = c(Q,M)C^{-1}(Q,M)$$

$$-a_I = s(Q,M)S^{-1}(Q,M) \tag{34}$$

ou

$$c_k(Q,M) = \sum_{\ell=-M}^{M}\sum_{j=-N_o}^{N_o} z(j) \begin{matrix}\cos\\ \sin\end{matrix}\left[2\pi\sigma_k\left[j+\ell(Q+2N_o+1)\right]\right]$$

$s_k(Q,M)$

Nous pouvons obtenir à l'aide de la méthode de Banachiewicz-Cholesky-Crout les matrices A(Q,M) et B(Q,M) équivalant à $C^{-1}(Q,M)$ et $S^{-1}(Q,M)$.

Les observations discontinues sur les pleines et basses mers peuvent donc être traitées à l'aide de la condition des moindres carrés. Ceci nous est très utile puisque l'on observe la marée dans le haut Saint-Laurent que durant la saison de navigation.

L'appendice 3 donne les matrices 2A(Q,M) et 2B(Q,M) qui correspondent à $2N_o + 1=177$, Q=175 2M+ 1=19. n=43 dans ce cas puisque le constituent (0,0,0,1) ne peut pas être distingué de (0,0,0,0) par un tel échantillonnage.

8.- Les Résultats pour Québec et Trois-Rivières.

Les Tables 1 et 2 donnent les résultats de l'analyse pour Québec et Trois-Rivières. Nous disposons de données continues pour Québec et de données discontinues pour Trois-Rivières. Nous avons jugé bon d'analyser Québec à l'aide des programmes pour les données continues et discontinues. De cette façon nous pouvons vérifier l'efficacité de la procédure. L'erreur quadratique est moindre pour les données obtenues durant l'intervalle novembre-mars, qui sont fortement perturbées pas les tempêts d'hiver. Il y a donc un certain avantage à ne pas inclure ces données dans l'analyse.

La Fig. 5 montre la différence entre l'observation et la valeur donnée par l'analyse pour les années échantillons 1946-1955-1965; nous notons que cette différence est maximum en hiver. De plus le moindre succès est obtenu pour la hauteur de la basse mer. Les données sur cette variable sont fortement irrégulières, ce qui indique que le fleuve lui-même n'est pas certain de ce que sera la hauteur de sa basse mer. Les pleines mers sont adéquatement représentées par l'analyse. La Table 3 donne sous forme statistique la différence entre les valeurs prédites et les valeurs observées pour l'année 1965 à Québec.

La Fig. 8 montre la filtrat $z_o(j)$ pour Québec sur l'intervalle d'observations. Nous notons une tendance nette des niveaux, surtout ceux de la basse mer à diminuer avec le temps. Les oscillations de longue période sont à peine perceptibles et très irrégulières; il n'est donc pas sage de les rechercher a l'aide de constituents.

Toutefois nous avons retenus les constituents de basse fréquence dans l'analyse pour évaluer le succès du filtrage.

Table 1

Résultat de l'analyse de 19 années d'observations continues à Québec

Les valeurs en parenthèses indiquent les valeurs analysées après avoir enlevé le filtrat passe-bas. L'erreur probable est indiquée seulement pour la première série de valeurs. L'erreur probable du module est la même pour toutes les ondes, soit $\pm$.1 min.

Table 1
5 hr. 43 min. 22/12/55

Nadir

T=3

	Basse Mer									Pleine Mer					
	Temps					Hauteur				Temps			Hauteur		
		Min		Deg			Pieds	Deg		Min	Deg		Pieds	Deg
0	(.0)	76.5	±.1	-----		(.00)	1.52	-----	(.0)	375.0	-----	(.00)	15.19	-----
1	(.0)	1.7		144.2		(.00)	.22	50.4	(.0)	1.4	346.8	(.00)	.18	144.3
2	(.0)	1.4		148.4		(.01)	.41	32.7	(.1)	2.4	229.8	(.01)	.35	61.0
3	(.0)	.5		85.2		(.00)	.05	336.3	(.1)	1.2	128.7	(.00)	.05	20.8
4	(.2)	1.7		129.4		(.03)	.26	79.0	(.2)	1.3	237.6	(.03)	.28	71.5
5		1.1		19.4	±4.1		.08	53.5		.7	202.9		.14	279.6
6		9.4		129.0	.9		.08	7.4		8.6	117.6		.08	176.9
7		.0		90.7	----		.05	126.0		.4	11.1		.06	134.6
8		7.3		154.2	.6		.32	198.4		12.6	181.8		.89	24.7
9		.2		205.3	22.3		.02	109.0		.3	13.9		.01	178.6
10		21.5		278.0	1.4		.22	114.6		9.2	285.1		1.47	97.4
11		4.2		282.4	9.1		.05	65.3		1.8	263.3		.24	99.4
12		.3		91.8	16.0		.03	228.2		.4	85.8		.05	196.0
13		.8		127.9	5.9		.03	325.3		.6	122.8		.03	296.4
14		2.7		36.1	1.7		.08	283.6		2.4	15.5		.08	109.6
15		2.0		336.1	2.3		.00	48.2		1.7	317.0		.06	33.2
16		38.9		327.9	.2		.53	211.9		37.4	301.8		1.72	35.2
17		.3		293.1	13.2		.01	58.7		.4	308.6		.02	241.6
18		1.5		290.6	2.9		.05	164.4		.7	315.3		.04	108.1
19		.3		167.9	16.0		.03	356.6		.4	222.7		.07	7.1
20		1.2		295.1	3.6		.01	296.5		.6	279.7		.13	117.3
21		.2		175.8	17.8		.01	88.5		.2	167.0		.04	20.1
22		10.7		22.1	.4		.14	205.0		6.7	10.9		.41	37.3
23		3.7		37.1	1.1		.06	228.8		2.3	41.3		.11	27.6
24		3.0		344.0	1.7		.04	237.9		2.6	318.9		.09	53.8
25		.2		113.5	22.8		.03	105.7		.3	9.9		.03	78.3
26		.2		41.2	22.3		.01	351.4		.2	299.1		.01	65.3
27		1.8		340.9	2.8		.01	96.7		1.5	321.6		.15	55.2
28		.1		210.3	----		.02	232.1		.1	349.8		.01	313.6
29		2.9		138.8	1.7		.14	4.8		2.1	170.0		.06	260.7
30		1.6		.3	2.6		.04	274.9		1.4	339.5		.03	34.9
31		.1		249.1	----		.01	55.6		.2	240.5		.02	30.2
32		1.5		41.1	3.1		.01	78.4		1.1	39.6		.06	54.7
33		.9		251.8	7.2		.06	153.2		.9	246.6		.01	6.8
34		6.3		187.8	.6		.12	73.4		6.0	165.8		.16	255.1
35		.2		227.0	25.8		.02	95.9		.3	87.5		.04	83.6
36		3.6		188.0	1.1		.07	79.5		3.6	163.5		.06	248.3
37		.4		225.4	12.9		.01	168.0		.3	195.3		.04	223.1
38		.8		195.9	5.1		.02	104.0		1.0	188.3		.02	240.5
39		.5		208.6	8.9		.01	326.2		.6	179.3		.02	244.1
40		.7		351.1	7.2		.03	234.8		.5	47.7		.01	98.6
41		.5		183.5	8.6		.02	297.0		.4	166.6		.05	285.2
42		1.2		49.0	3.6		.03	261.0		1.6	25.7		.03	146.4
43		1.0		47.2	4.2		.02	303.8		1.0	29.5		.03	76.0
44		.3		250.9	15.6		.01	163.3		.3	269.9		.01	323.3

Table 1(a)
18 hr. 6 min. 22/12/55

141 | Zenith

		Basse Mer						Pleine Mer				
		Temps			Hauteur			Temps			Hauteur	
		Min	Deg		Pieds	Deg		Min	Deg		Pieds	Deg
0	(.0)	76.6	-----	(.00)	1.51	-----	(.0)	374.8	-----	(.01)	15.20	147.7
1	(.0)	1.8	137.0	(.00)	.23	45.2	(.0)	1.5	347.6	(.00)	.20	147.7
2	(.1)	1.7	142.9	(.01)	.40	32.1	(.1)	2.5	229.6	(.01)	.34	65.0
3	(.1)	.5	109.9	(.00)	.03	341.1	(.1)	.9	133.9	(.01)	.05	.8
4	(.2)	1.5	124.8	(.03)	.26	80.5	(.2)	1.4	247.6	(.03)	.27	74.0
5		1.0	231.4		.03	202.2		1.0	2.8		.23	78.4
6		9.8	129.0		.09	5.2		8.5	117.9		.07	169.9
7		.3	246.3		.05	136.6		.3	78.0		.06	131.4
8		7.2	155.4		.33	199.7		12.3	181.8		.92	24.0
9		.1	147.3		.01	235.8		.3	287.3		.04	99.1
10		17.9	100.5		.20	287.0		6.7	113.5		1.51	277.0
11		3.1	97.2		.05	238.3		.8	51.1		.23	284.5
12		.2	257.3		.03	204.1		.2	75.9		.05	171.0
13		.6	133.8		.03	323.3		.4	99.7		.02	278.7
14		2.6	37.4		.08	284.7		2.4	12.7		.07	105.9
15		1.7	325.3		.01	322.2		1.6	317.1		.08	29.6
16		38.7	327.9		.53	211.1		37.5	302.4		1.72	35.6
17		.5	261.7		.02	108.9		.6	303.7		.02	144.0
18		.4	129.6		.02	345.3		.7	158.8		.04	304.7
19		.4	150.3		.03	344.1		.5	200.8		.07	1.3
20		1.3	125.2		.02	276.6		.7	139.6		.15	294.7
21		.6	2.2		.01	187.3		.4	60.4		.03	160.0
22		10.3	21.7		.15	208.4		6.4	8.8		.44	37.1
23		3.7	39.9		.06	233.0		2.3	43.0		.13	19.6
24		2.9	347.1		.05	233.8		2.6	323.8		.09	48.1
25		.4	63.2		.01	137.5		.3	42.9		.04	83.0
26		.2	17.8		.00	165.1		.2	331.2		.01	41.2
27		1.9	341.1		.02	130.0		1.6	315.5		.15	46.0
28		.2	66.1		.02	276.4		.1	179.3		.03	224.8
29		2.3	331.8		.12	192.0		2.2	1.5		.02	96.2
30		1.4	358.3		.03	273.7		1.4	334.0		.04	22.7
31		.1	350.5		.00	34.1		.3	307.9		.01	110.1
32		1.5	44.8		.01	211.4		1.3	36.9		.05	47.8
33		1.0	259.7		.05	157.7		.7	236.0		.02	144.8
34		6.4	189.9		.12	78.9		5.7	166.5		.13	258.1
35		.2	243.7		.03	106.8		.2	356.6		.04	96.0
36		3.3	183.0		.06	79.0		3.3	163.2		.08	241.8
37		.7	213.2		.01	240.3		.4	191.6		.02	227.2
38		1.0	202.5		.02	129.3		.7	184.0		.02	263.2
39		.6	203.2		.01	262.1		.7	174.8		.04	284.1
40		.5	185.4		.04	45.2		.7	213.9		.01	326.0
41		.5	188.9		.03	320.3		.5	163.4		.05	269.8
42		1.3	42.0		.02	285.1		1.3	24.1		.05	140.2
43		1.0	42.6		.01	308.3		1.1	18.1		.03	70.0
44		.2	269.8		.01	63.8		.4	221.9		.00	324.2

Table 2

Résultat de l'analyse de 19 années d'observations discontinues à Grondines

Table 2
10 hr. 5 min 2/8/44

T=3 1290

	Basse Mer		Nadir		Pleine Mer			
	Temps		Hauteur		Temps		Hauteur	
	Min	Deg	Pieds	Deg.	Min	Deg	Pieds	Deg.
0	272.8	-----	4.56	-----	506.6	-----	10.85	-----
1	-----	-----	----	-----	-----	-----	-----	-----
2	8.9	20.8	1.47	1.8	6.0	199.0	.90	3.3
3	1.4	20.8	.23	311.1	.6	114.3	.15	316.7
4	3.2	72.6	.61	1.0	.8	297.7	.34	1.2
5	1.8	43.0	.06	25.9	1.5	179.5	.09	261.6
6	8.3	119.1	.04	291.7	8.0	111.7	.08	198.6
7	.5	351.3	.04	48.0	.4	292.2	.03	62.7
8	9.8	167.0	.22	61.0	11.9	185.4	.67	30.9
9	.7	345.7	.04	13.7	.6	333.8	.03	75.0
10	25.3	278.4	.09	274.6	11.0	282.6	.93	97.6
11	4.6	276.3	.01	166.9	2.2	256.2	.17	103.2
12	.4	41.7	.03	247.2	.7	5.6	.06	202.5
13	.9	128.5	.01	220.4	.5	111.0	.02	262.2
14	1.4	6.5	.11	204.0	1.4	2.8	.05	86.3
15	1.4	311.0	.00	48.2	1.6	294.1	.05	36.8
16	34.9	317.5	.50	83.8	38.3	301.3	1.22	42.0
17	.2	8.4	.03	245.4	.1	89.5	.04	250.4
18	1.3	285.7	.02	265.2	.7	284.6	.02	118.6
19	.8	213.7	.01	14.2	.8	211.2	.04	29.5
20	1.2	295.4	.02	227.9	.2	322.4	.07	127.9
21	.3	39.9	.02	88.6	.2	289.5	.02	56.0
22	8.9	20.2	.09	57.2	6.1	358.6	.28	39.2
23	3.3	41.1	.03	54.6	2.2	43.1	.10	39.9
24	2.7	335.1	.01	168.1	2.4	317.3	.06	72.4
25	.3	158.5	.01	110.7	.4	105.1	.01	74.9
26	.3	23.4	.01	315.2	.1	3.2	.01	298.5
27	1.5	353.1	.04	51.4	1.7	341.8	.07	38.5
28	.4	256.5	.01	6.6	.3	313.6	.02	40.1
29	3.2	125.5	.03	36.2	2.5	145.9	.07	304.0
30	1.3	337.9	.00	54.4	1.2	319.4	.01	20.2
31	.2	102.3	.00	321.5	.2	96.9	.01	7.3
32	1.5	51.2	.03	54.6	.9	61.4	.05	53.1
33	.9	227.2	.00	180.6	.6	211.4	.02	326.4
34	5.7	176.2	.05	344.3	5.9	163.9	.13	256.2
35	.1	324.3	.01	82.1	.2	249.3	.02	76.8
36	2.9	171.1	.02	309.7	3.4	155.4	.07	249.8
37	.4	193.9	.01	282.4	.5	178.4	.02	234.7
38	.8	169.8	.01	34.1	.9	158.6	.01	303.7
39	.4	215.6	.01	343.4	.4	208.2	.01	303.7
40	.6	323.4	.01	235.3	.6	357.3	.01	213.6
41	.3	222.4	.01	297.2	.4	140.4	.01	229.8
42	1.1	24.6	.00	274.1	1.4	4.8	.02	69.4
43	.9	25.2	.00	169.2	1.0	10.2	.02	95.3
44	.2	223.1	.01	341.0	.5	195.6	.01	261.0

Table 2 (a)
22 hr. 35 min. 2/8/44

	Basse Mer Temps		Zénith Hauteur		Pleine Mer Temps		Hauteur	
	Min	Deg	Pieds	Deg.	Min	Deg	Pieds	Deg
0	273.8	-----	4.60	-----	505.2	-----	10.86	-----
1	-----	-----	-----	-----	-----	-----	-----	-----
2	9.6	14.1	1.47	.6	6.9	181.3	.90	2.5
3	1.1	23.6	.24	310.4	.8	110.3	.15	317.2
4	2.9	61.9	.60	358.6	.9	339.2	.32	358.5
5	1.9	232.7	.03	236.5	1.7	341.7	.12	71.8
6	8.2	120.0	.04	289.2	7.9	111.2	.07	199.9
7	.1	320.9	.03	60.5	.0	299.2	.03	28.9
8	9.6	167.3	.22	58.4	12.2	184.1	.66	31.2
9	.6	91.6	.04	12.6	.5	146.1	.02	10.2
10	21.9	102.4	.11	112.0	8.2	109.8	.94	275.9
11	3.9	107.7	.05	151.6	1.9	89.5	.17	276.1
12	.2	128.0	.03	264.6	.1	100.1	.05	208.9
13	.4	104.3	.01	252.0	.5	93.4	.02	176.5
14	1.6	1.9	.10	212.8	1.1	14.1	.05	79.2
15	1.3	336.5	.01	23.1	1.3	299.8	.04	38.4
16	35.1	317.2	.48	83.7	38.0	301.9	1.22	42.1
17	.4	269.9	.04	242.0	.3	346.1	.03	277.0
18	.9	109.4	.03	310.7	1.3	148.3	.04	268.6
19	.6	181.6	.01	43.5	.5	231.5	.04	356.0
20	1.2	123.8	.02	182.6	.1	67.1	.05	275.3
21	.4	30.8	.01	83.7	.2	19.7	.02	89.2
22	8.6	19.3	.08	47.7	6.0	2.2	.30	38.7
23	3.1	44.3	.04	64.3	2.1	42.1	.11	29.8
24	2.4	335.0	.02	173.7	2.4	320.2	.05	61.7
25	.5	98.9	.02	152.7	.6	27.6	.02	125.6
26	.2	71.3	.01	324.7	.2	1.2	.01	333.5
27	1.7	354.8	.05	64.5	1.9	329.9	.08	50.0
28	.4	108.2	.00	281.6	.1	224.2	.02	317.7
29	2.7	326.1	.03	182.1	2.7	349.3	.09	116.3
30	1.0	325.0	.01	39.3	1.1	312.8	.03	23.5
31	.2	20.2	.00	60.3	.1	290.3	.00	150.5
32	1.5	48.3	.03	76.4	1.2	35.9	.04	48.3
33	.5	236.4	.01	53.0	.3	252.0	.02	257.4
34	5.5	175.8	.05	339.7	5.4	163.0	.13	260.4
35	.3	32.6	.01	114.3	.1	35.2	.01	136.7
36	3.0	173.8	.02	297.2	3.0	153.2	.07	249.0
37	.3	223.5	.02	275.5	.4	198.6	.02	234.0
38	.8	182.2	.01	60.8	.7	194.0	.01	281.8
39	.8	178.7	.01	350.9	.7	189.1	.02	305.8
40	.7	191.5	.01	345.0	.9	186.1	.02	326.5
41	.4	176.1	.01	288.9	.5	164.9	.01	251.9
42	1.1	32.5	.01	26.9	1.0	14.1	.02	95.3
43	.8	34.8	.00	126.2	.7	16.5	.02	86.3
44	.2	276.9	.01	259.7	.3	213.5	.02	291.1

Fig. 5. Différence entre la valeur observée et la valeur prédite pour trois années échantillons à Québec.

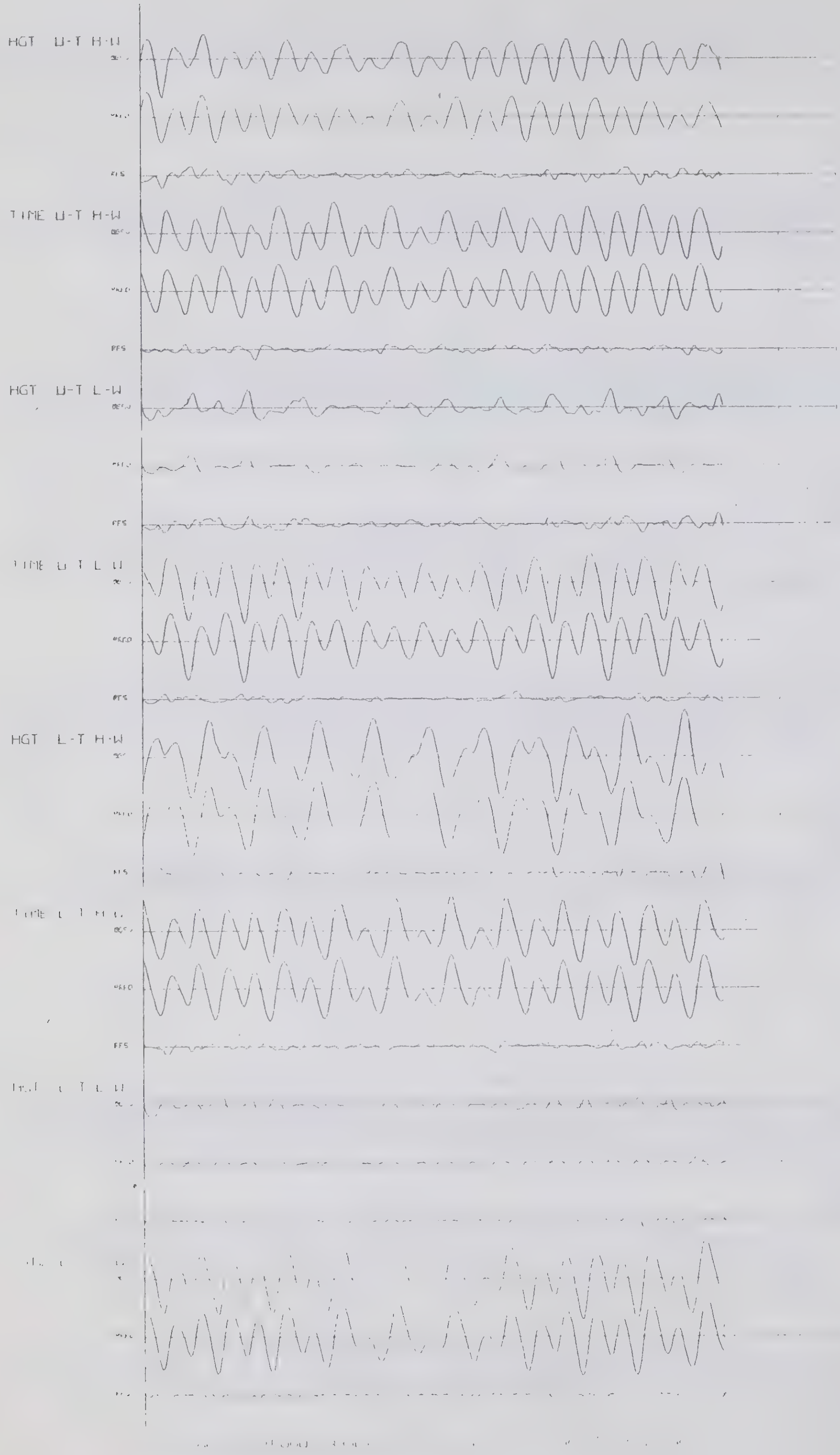


Fig. 5 (suite)

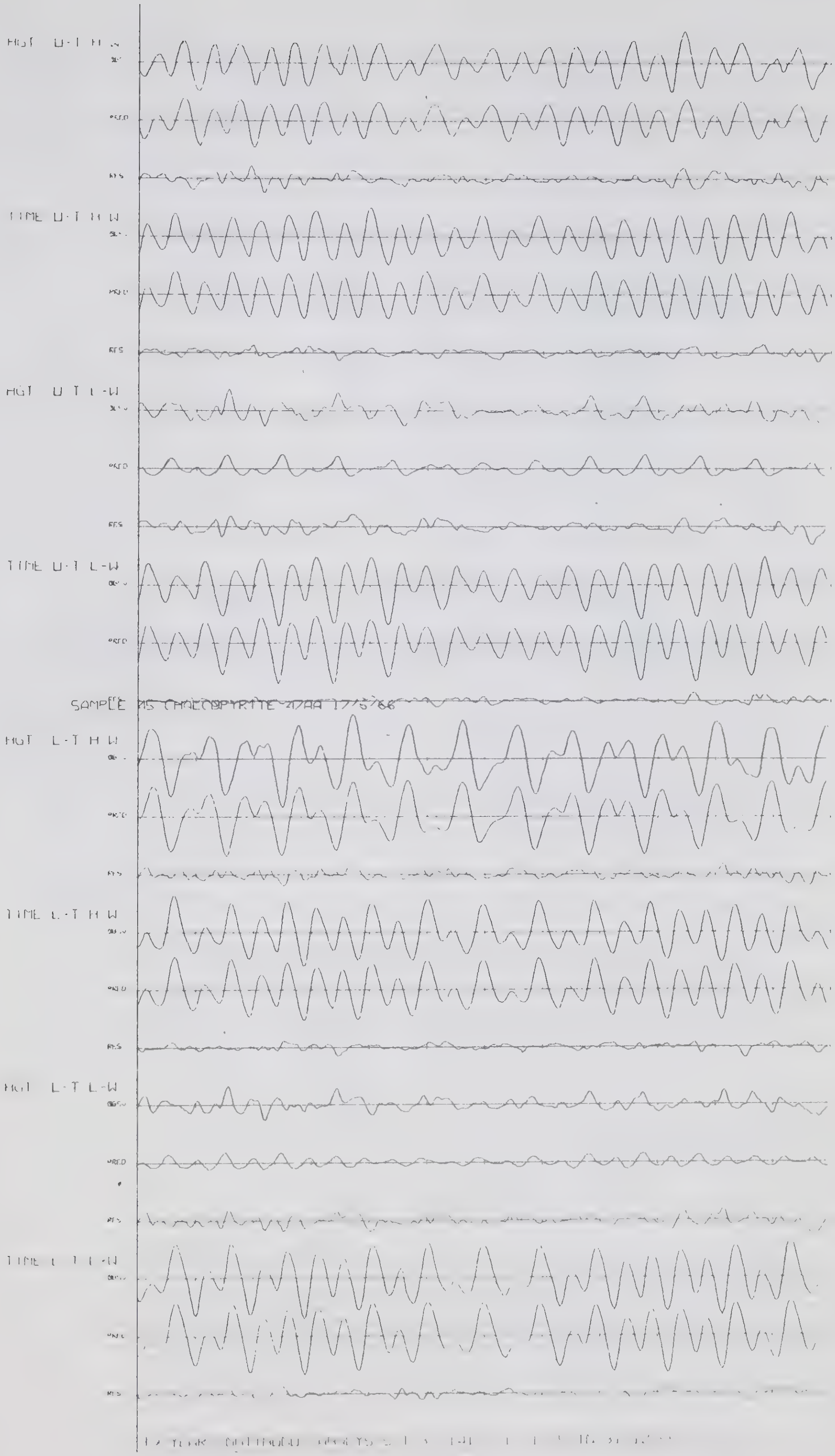
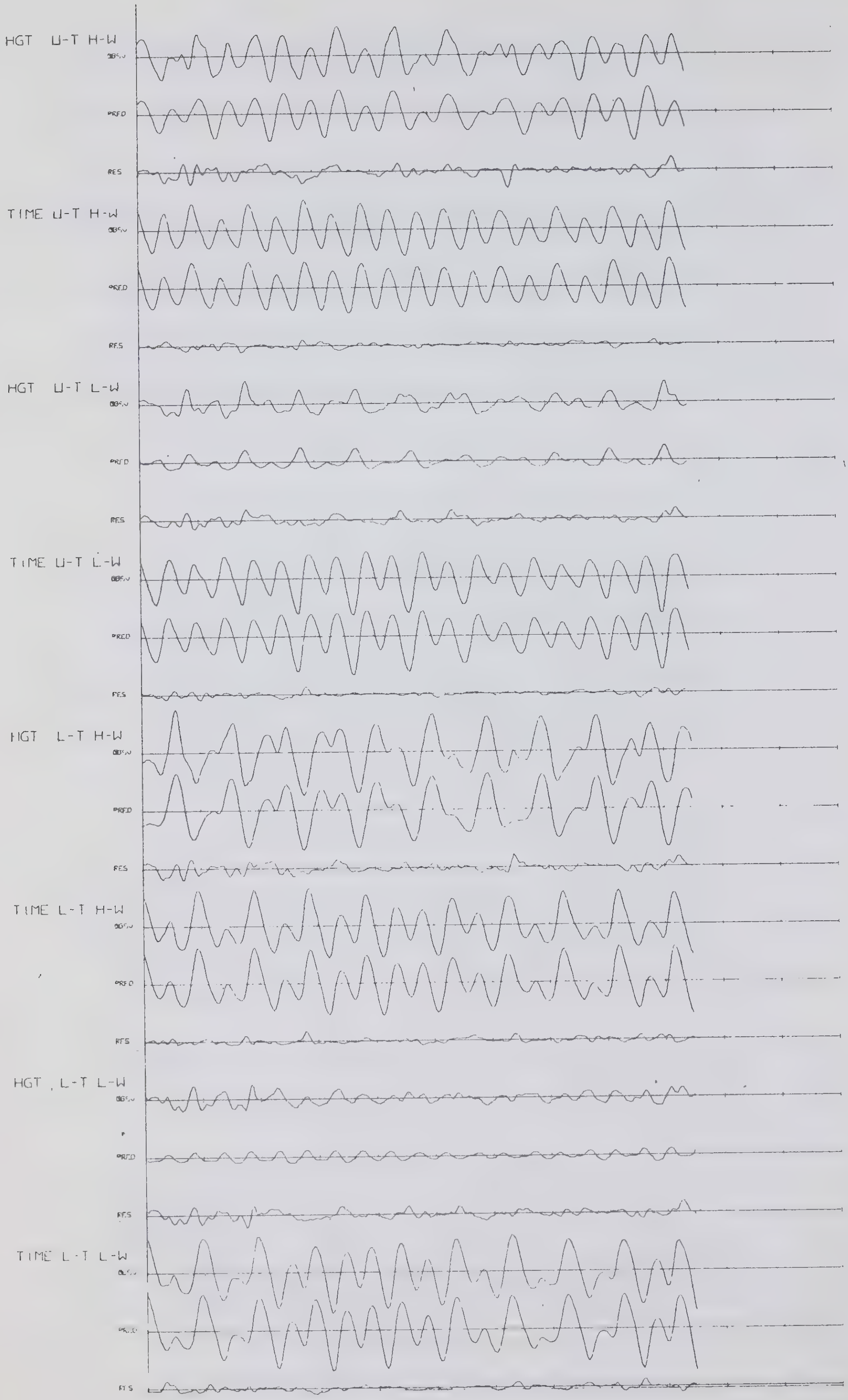


Fig. 5 (suite)



19 YEAR CONTINUOUS ANALYSIS T 3 (4) 1/ 1/65 TO 16/10/65

Table 3

Table Statistique des différences entre les valeurs des pleines et basses mers observées et les valeurs prédites à l'aide de la méthode allemande pour les ports de Québec et Grondines durant l'année 1965. (Calculée par la Service Hydrographique)

Table 3

Québec

Différences de temps en minutes

Pleines Mers 0 à 10	11 à 20	21 à 30	plus que 30	Moyenne	Max	Mois	Basses Mers 0 à 10	11 à 20	21 à 30	plus que 30	Moyenne	Max
48	9	3	0	-3.6	20-25	Jan	46	11	2	1	0.2	40-19
38	13	3	0	-2.3	27-30	Fév	40	10	3	1	1.5	28-33
55	2	3	0	-0.5	25-20	Mars	51	7	0	2	0.6	36-20
52	5	1	0	-0.3	20-22	Avr	55	3	0	0	0.8	15-10
59	1	0	0	-0.6	10-15	Mai	59	1	0	0	-0.4	8-12
52	6	0	0	4.1	19-10	Juin	56	2	0	0	0.1	12-13
58	2	0	0	2.5	20-10	Juil	53	7	0	0	4.4	20-10
58	2	0	0	1.1	15-20	Août	57	3	0	0	3.9	18-7
51	7	0	0	-1.0	15-15	Sept	49	8	0	1	3.4	32-15
50	10	0	0	-4.3	14-16	Oct	54	6	0	0	2.6	20-20
49	8	1	0	-0.9	25-15	Nov	43	13	1	1	2.5	35-18
51	8	0	0	-6.3	5-20	Déc	57	3	0	0	0.2	13-15
621	73	11	0	-1.0	27-30	Année	620	74	6	6	1.6	40-33

Table 3(b)

Différences de hauteur en pieds

.0 à .5	.6 à 1.0	1.1 à 1.5	plus que 1.5	Moyenne	Max		.0 à .6	.6 à 1.0	1.1 à 1.6	plusque 1.5	Moyenne	Max
21	18	10	11	-0.2	1.8-4.4	Jan	36	12	11	11	0.2	1.8-1.5
24	10	12	8	-0.1	3.8-2.5	Fév	36	14	1	3	-0.1	3.9-1.6
30	19	4	7	-0.3	1.0-3.4	Mars	31	21	7	1	-0.4	0.8-2.6
38	13	7	0	-0.3	1.5-1.5	Avril	22	31	5	0	-0.5	1.4-1.2
38	17	5	0	-0.3	1.4-1.3	Mai	27	27	6	0	-0.5	1.4-1.3
43	10	5	0	-0.2	1.3-1.4	Juin	33	20	5	0	-0.4	1.4-1.3
46	6	2	6	-0.1	1.8-2.9	Juil	45	15	0	0	-0.3	0.7-1.0
49	11	0	0	0.0	0.8-0.9	Août	52	8	0	0	0.1	0.8-0.8
37	16	3	2	0.0	1.6-2.3	Sept	48	8	2	0	0.2	1.1-1.1
25	16	8	11	0.5	2.8-2.1	Oct	27	15	9	9	0.7	2.5-1.1
18	16	9	15	0.8	4.5-1.9	Nov	23	19	5	11	0.8	4.3-0.8
20	16	15	8	0.3	1.9-2.3	Déc	34	12	11	3	0.5	2.2-0.9
389	168	80	68	0.0	4.5-4.4	Année	414	202	62	28	0.0	4.3-2.6

Table 3(c)

Grondines

Différences de temps en minutes

0 à 10	11 à 20	21 à 30	plus que 30	Moyenne	Max		0 à 10	11 à 20	21 à 30	plus que 30	Moyenne	Max
20	25	8	7	15.6	44-15	Jan	31	19	8	2	-7.2	30-55
19	15	7	2	12.2	42-23	Fév	14	15	7	7	-16.0	30-55
31	25	3	1	9.7	35-19	Mars	13	19	20	8	-20.9	-5-55
49	8	1	0	2.8	22-10	Avril	35	19	3	1	-5.1	23-25
49	10	1	0	-4.7	8-21	Mai	56	4	0	0	-1.2	14-16
54	4	0	0	-1.1	10-19	Juin	44	13	1	0	-7.3	4-22
55	5	0	0	-2.2	10-18	Juil	56	3	1	0	-4.3	9-25
55	4	0	0	-3.7	10-20	Août	45	14	1	0	5.2	30-20
22	36	0	0	-12.7	15-20	Sept	40	15	1	2	7.4	35-10
10	12	15	23	3.6	64-45	Oct	14	9	7	30	41.1	115-9
36	13	7	2	-11.4	15-35	Nov	17	19	12	10	18.8	44-7
40	19	0	1	-3.8	40-20	Déc	24	19	13	4	11.9	68-25
440	176	42	36	0.4	64-45	Année	389	168	74	64	1.9	115-55

Table 3(d)

Différences de hauteur en pieds

.0 à .5	.6 à 1.0	1.1 à 1.5	plus que 1.5	Moyenne	Max		.0 à .5	.6 à 1.0	1.1 à 1.5	plus que 1.5	Moyenne	Max
28	13	6	13	-0.7	1.6-3.5	Jan	19	15	8	18	-1.0	0.9-3.2
10	9	10	14	-1.1	2.4-3.2	Fév	2	3	5	33	-2.1	1.4-3.7
17	13	15	15	-1.0	0.6-3.6	Mars	4	8	4	44	-2.2	-0.3-3.9
35	10	5	8	-0.5	1.0-2.3	Avr	20	15	2	21	-1.1	0.7-3.3
44	12	3	1	0.2	1.6-0.8	Mai	37	19	4	0	0.2	1.5-0.7
48	10	0	0	-0.1	0.9-1.0	Juin	36	19	3	0	-0.3	0.8-1.2
51	9	0	0	-0.2	0.6-0.8	Juil	53	7	0	0	-0.3	0.6-0.9
47	12	0	0	0.0	1.0-0.7	Août	38	20	2	0	0.2	1.2-0.8
46	10	1	1	0.1	1.1-1.6	Sept	27	30	1	0	0.5	1.2-0.7
10	17	14	19	1.1	2.6-0.9	Oct	0	4	6	50	2.2	3.4,0.7
6	4	22	26	1.5	4.0-0.3	Nov	0	1	2	55	2.4	4.2,0.9
17	15	17	11	0.7	2.3-1.4	Déc	11	8	7	34	1.5	2.8-0.4
359	134	93	108	0.0	4.0-3.6	Année	247	149	4	255	0.0	4.2-3.9

Fig. 6. Le résultat du filtrage passe-bas à Québec.

BIBLIOGRAPHIE

Godin G. 1966. Le Niveau Moyen Journalier et les Seiches de Courtes Périodes. p. 77-92. Revue Hydrographique Internationale. Vol. XLIII No. 2.

Godin G. 1966a. The Analysis of Tides Par. 2.4 230 p. Sera bientôt publié.

Horn. W. 1960. Some Recent Approaches to Tidal Problems. International Hydrographic Review. 65-84. Vol. XXXVII, No. 2.

Korn G.A. & Korn T.M. 1961. Mathematical Handbook for Scientists and Engineers. 636-943.p. McGraw Hill Book Co. Inc.

II

Détails Pratiques

(Specifications.)

Cette section donne le contenu des instructions transmises à M. J. Taylor pour l'élaboration des programmes. Le lecteur doit noter que les instructions n'ont pas été suivies à la lettre dans les programmes et qu'il y a certaines modification de détails.

Specifications for the Analysis of 19
Years of Continuous Observations on the
Time and the Height of High and Low Water.

Input Card Format

TCxssssxddxmmyyxttttxhh.hxttttxhh.hxttttxhh.hxttttxhh.(cont'd)
hxDDDxDDDxDDDxDDDxxxxxcccc

C: code = 1,2,3,4,5,6,7,8
ssss: Station number
x: blank or area for sign
dd: day number
mm: month number
yy: two last digits of year number
tttt: time of occurrence of extremum in hours and minutes
hh.h: height of the extremum in feet given to the tenth of a foot. It may be negative at times.
xDDD: time difference in minutes between the occurrence of the extremum and the lunar transit. This may be positive or negative.
cccc: card counter

General Summary

1. Compute a table of cosθ for $.0000 \leq \theta \leq 1.0000$ in steps of $\Delta\theta = .0001$ to 4 decimal places. The angles are measured in fractions of 2π.

2. Read the first input card.
Transfer TCxssss into the output area as heading.

3. Set up a data counter. There will be 6689 data organized into 3 blocks:

   3344,1,3344.

   In one data there will be 8 individual fields:

   xDDDxhh.hxDDDxhh.hxDDDxhh.hxDDDxhh.h
   1 2 3 4 5 6 7 8

4. Read in the 3344 data into their proper area following code C to fill in the first data. This involves reading each card trying to fill up the data with the eight required fields. One card may not suffice to do that; this occurs when an extremum is missing on a given day. In this case the field xttttxhh.h on the card is replaced by x9999x99.9 .

5. On the card (or cards) corresponding to data no. 3345 transfer to the output area the 2 fields

ddxmmyytttt

corresponding to the 2 pairs of fields xDDDxhh.hxDDDxhh.h . Then transform xDDD into hours and minutes and subtract it from tttt. One obtains then in the output area the field

$$\mathrm{ddxmmyy}t_o t_o t_o t_o$$

which indicate the time of the lunar transit corresponding to the time and height of the extremum given in the field xDDDxhh.h. There should be two such times of lunar transit in the output area. Note the address of data no.3345.

6. Go on filling the data area with the remaining 3344 data. If short of cards send an error message and terminate the program.

7. Remembering that in each data there are 8 distinct fields to be manipulated, get to data 3345. Call it Y_o so that the ith data to the right will be y_i and the ith data to the left will be y_{-i}.

8. Put y_{-i} into buffer area. Calculate y_i+y_{-i}. Put it over y_i. Calculate y_i-y_{-i}. Do this for i=1,2,3,...,3344. At the end of this cycle one has calculated

$$y_i \pm y_{-i}$$

9. Clear the column vector area $C_o, C_1, \ldots, C_{44}, S_1, S_2, \ldots, S_{44}$.

This area will be used to accumulate sums of products which will become equal to the desired column vectors in the last cycle.

10. Calculate

$$y_o + \sum_{i=1}^{i=3344} (y_i + y_{-1}) \quad \text{in } C_o$$

11. Add y_o to C_k for k=1,2,3,....,44.

12. Read in the σk cards (angular speeds). Set up 2x44 accumulating areas for

$$\sigma_k i \quad \text{and} \quad \sigma_k i + 1/4$$

one will be used to compute the argument of the cosine, the other, the argument of the sine. Clear both areas. Add .2500000000 to the σk^i for the sine. This allows looking up the sine in the cos table.

13. Add σ_k to ten decimal places to both areas. Remove any excess 1.00 from the arguments. In a buffer area, transfer both arguments to 5 decimal places and round off to 4 places. Look up the cos of both arguments in the cos table.

14. Then compute $(y_i+y_{-i}) \cos \sigma k^i$ for each of the 8 fields in data i. Add this to C_k.

    Compute $(y_i-y_{-i}) \sin \sigma_k i$ for each of the 8 fields in data -i. Change the sign of the product and add the result to S_k.

    Do this in turn for k=1,2,3,....,44.

15. Do 13.- for i=1,2,3,...,3344. At the end of the last cycle one will have computed

$$C_k = \sum_{i=1}^{3344} (y_i+y_{-i})\cos \sigma k^i \text{ and } S_k = \sum_{i=1}^{3344} (y_i-y_{-i})\sin \sigma_k^i$$

    for k=1,2,3,...,44.

16. Read in the A and B matrices.

17. Compute $\sum_{k=0}^{44} A_{ok}C_k = X_o$ for each of the 8 fields within C_o. Put X_o over C_o. Go on to compute $\sum_{k=0}^{44} A_{ik}C_k = X_i$ and

$$\sum_{k=1}^{44} B_{ik}S_k = y_i \quad \text{for } i = 0,1,2,3,...,44$$

    wiping out C_k and S_k with X_i and Y_i in sequence.

18. Calculate

$$A_i = (X_i^2 + Y_i^2)^{1/2}$$

    and $\qquad$ i=1,2,3,...,44.

$$a_i = \text{arc tan } (Y_i/X_i)$$

19. Read in the astronomical arguments matrix.

20. Calculate with the help of this matrix and the fields

    ddxmmyy$t_o t_o t_o t_o$, V_i

    for each pair of fields xDDDxhh.hxDDDxhh.h. V_i is the astronomical argument which indicates the true phase of each constituent measured in Greenwich time. In order to calculate V_i one must be familiar with the absurdities of the Gregorian calendar.

21. Calculate $g_i = a_i + V_i$ $i = 1,2,3,\ldots,44$. the Greenwich phase lag.

22. The analysis is finished; one must output the result in a convenient format.

For instance

ANALYSIS OF 19 YEARS OF CONTINUOUS OBSERVATIONS
ON THE TIME AND THE HEIGHT OF HIGH AND LOW WATER

TCxssss

ddxmmyy$t_o t_o t_o t_o$

ddxmmyy$t_o t_o t_o t_o$

Upper Transit — Lower Transit

	Low water			High water		Low water		High water	
	Time	Diff.	Height	Time	Diff.Height	Time	Diff.Height	Time	Diff.Height
00	A_o	A_o	A_o	A_o	A_o	A_o	A_o	A_o	
01	A_1,g_1	A_1,g_1	A_1,g_1	A_1,g_1	A_1,g_1	A_1,g;	A_1,g_1	A_1,g_1	
02									
.									
.									
.									
44									

The above is an example for C=1.

DETAILS

The cosine Table

When angles are quoted, they are in fractions of 2π. This is the most natural way of computing with angles although this does not seem at all understood in computing circles.

1. One has to insert into the table

$$\cos .0000 = .9999$$
$$\cos 1.0000 = .9999$$

in order to avoid the necessity of carrying a figure in the unity column.

2. One may compute the table only for values of θ lying between .0001 and .2500 and program to fill in the area for θ = .2500 to .9999, using the relations

$$\cos(.2500 + .x) = -\cos(.2500 - .x) \qquad .0000 \leqslant .x \leqslant .2500$$

$$\cos(.5000 + .x) = -\cos .x \qquad .0000 \leqslant .x \leqslant .5000$$

3. Since the subroutine for the evaluation of $\cos\theta$ requires θ to be in radians, one may calculate for θ in radians at all time or translate θ in fractions of 2π into radians at each step. Thus

a) $\Delta\theta = .0001 \rightarrow \quad \theta = .002\pi$

b) $\Delta\theta = .0001 \qquad \theta = n\Delta\theta$ translated into radians at each step.

In a) there is a very serious danger of error build up unless one carries θ to a respectable number of figures. There is no such danger for b) where the increment is .0001 exactly.

Once the table is completed, it should be listed with respect to the decimal values of the angle.

4. It is possible to store only the table for arguments between .0000 and .2500. In this case the evaluation of $\cos\theta$ and $\sin\theta$ for any angle θ requires more sophisticated programming. If space allows, it is best to store the table for θ between .0000 and 1.0000 (included).

The Input Deck

There should be about 7000 cards in that deck. They are naturally all identical in format. The first card of the deck should be used to abstract

TCxssss

which gives the number of the station (identification) and C the code number which indicate the order in which the high and low waters are organized with respect to the upper and lower lunar transit and how the first data should be filled in.

On any card the field xttttxhh.h could be replaced by the field x9999x99.9. This indicates that one extremum is missing on that day (nothing wrong with that. It just underlines the fact that we measure time with the sun while the tide moves with the moon; and a lunar day is longer than a solar day by 50 minutes). There will be as many x9999x99.9 fields as there are missing extrema. When a field indicating a missing extremum is sensed, the computer should move to the next extremum on the same card or move to the next card if the card has been completely scrutinized. On a given card the first xDDD field corresponds to the first xTTTT field and so on.

The Data Build-up

By reading the input deck one must build up a set of 6689 data. One data consists of eight consecutive fields read off from one or more input cards. One data consists of the following sequence of fields:

xDDDxhh.hxDDDxhh.hxDDDxhh.hxDDDxhh.h

These will be two high water and two low water times and heights corresponding to one complete rotation of the moon around the earth. Each of these fields is considered as a distinct variable to be analyzed but the process of analysis is identical for all of them. They are therefore handled as one unit although any operation on one data involves eight distinct but identical operations on the eight fields to be found in one data.

The xTTTT field on the input card is not used in the analysis but rather xDDD which gives in minutes the difference (positive or negative) between the time of occurrence of the extremum (xTTTT) and the time of the corresponding lunar transit.

To fill in the first data, one must look up the code. For C=1,3,6 and 8, one picks the first field on the data card to start filling in the first data.

For C=2,4,5 and 7, one must ignore the first field on the data card and start filling in the first data with the second field xDDDxhh.h. This simply means that the first extremum is ignored in order to reduce the number of possible combinations of lower and upper transits with high and low waters.

Once code C has been taken into account, one goes on filling the data in a continuous fashion, being however careful on the way to jump over the missing extrema.

After having accumulated 3344 such data, one must follow a special procedure in handling data no. 3345 (which might be spread on more than one card). This data is the central data and everything is centered around it. One must first transfer the first and third fields xddxmmyyxtttt corresponding to the two pairs of fields

xDDDxhh.hxDDDxhh.h

in data no. 3345 to the output area. Then one must translate the two xDDD into hours and minutes and subtract these new xDDD from xTTTT. One obtains then in the output area two fields of the form

$xddxmmyyxt_ot_ot_ot_o$

which give the time of upper and lower lunar transit to which the two pairs of fields

xDDDxhh.hxDDDxhh.h

in data no. 3345 are related. These times will be used subsequently in the calculation of the Greenwich astronomical argument V which will be used to calculate the Greenwich phase lag g.

Then one proceeds to fill data no. 3345 with the 8 fields which belong to it.

Then one fills in the remaining 3344 data.

If one is short of input cards to get 6689 data, one sends out an error message and terminates the program at this stage.

It is best to think of the address of these 6689 data as labelled by the symbol i. Data 3345 corresponds to y_o, Data 3344 to y_{-1}, Data 3346 to y_1. An original data in general is denoted as $y_{\pm i}$.

The $y_i \pm y_{-i}$

Having located 0, one moves to i=1 and i=-1. One saves y_{-1}, transfers y_1 into y_{-1}, then uses y_{-1} in storage to compute

$$y_1 + y_{-1} \text{ at } i=1$$

$$y_1 - y_{-1} \text{ at } i=-1$$

Then one moves to $i = \pm 2$ and repeats the same procedures. One does this for $i = \pm 1, \pm 2, \ldots, \pm 3344$. At the end of the last cycle, one has computed

$$y_i \pm y_{-i}$$

for i=1,2,3,..., 3344.

The Column Vector Area

One must set up in the computer an area for the location of the column vector components

$$c_o, c_1, \ldots, c_{44}, S_1, S_2, \ldots, S_{44}$$

Within each of these components, it must be understood that there must be room for the individual components corresponding to the eight fields present in each data. Thus:

$$C_o C_{o1} C_{o2} C_{o3} C_{o4} C_{o5} C_{o6} C_{o7} C_{o8}$$

In C_o, one computes (accumulates) the sum

$$y_o + \sum_{i=1}^{3344} (y_i + y_{-i}) = C_o$$

This is easily performed by adding all the fields to the right of y_o and including y_o in the sum. These sums are performed individually over the 8 fields making up one data. In this way $C_o C_{o1} C_{o2} \ldots C_{08}$ is filled up.

Then having properly cleared $C_1, C_2, , , \ldots, C_{44}$, one adds y_o to each of them. For instance, at the end of this cycle one should find in C_{23}

$$\underbrace{y_{o1} y_{o2} \cdots\cdots\cdots y_{o8}}_{C_{23}}$$

At this stage the calculation of C_o is completed. There is one y_o in all C_k for $k=1,2,3,\ldots,44$ and there is zero in all S_k, $k=1,2,3,\ldots,44$.

<u>The σ Cards</u>

This deck consists of 6 cards, each containing 8 fields 10 digits wide, with the exception of the last card which contains 4 fields. There are then 44 fields in all.

These fields give the value of the angular speed of the 44 constituents in fractions of 2π . On the left most digit of the field there is a flag (11 punch) which indicates the beginning of the field. The angular speed is given to 10 decimal places and one field stands for

.xxxxxxxxxx

with the decimal point <u>omitted</u>.

One has to carry that many decimal places because the argument $\sigma_k i$, although looked up only to four decimal places in the table, is calculated by incrementing repeatedly by σ_k which causes an appreciable error build up. This can be avoided only if one carries enough figures so that the error does not creep as far as the fourth decimal place.

By incrementing, a 1 is liable to appear to the left of the decimal point. In machines with overflow features, this digit is automatically dropped (automatic elimination of multiples of 2π). But in the actual computer one has to program to eliminate this digit.

The Calculation of the Column Vectors C and S

As a preliminary one must first set 2x44 accumulating areas for the calculation of

$$\sigma_k i \text{ and } \sigma_k i+1/4 \qquad k=1,2,3,\ldots,44$$

To evaluate sine $\sigma_k i$ we use the fact that

$$\cos(\sigma_k i+1/4) = -\sin \sigma_k i$$

Before setting the loop one must clear the area for

$$\sigma_k i \text{ and } \sigma_k i+1/4$$

then add .250000000 to each i+ 1/4 (44 of them)

Then one is ready to enter the loop.

Get the address of C_1 and S_1. Set i=i

Add each σ_k to its proper

$$\sigma_k i_o \quad \text{and} \quad \sigma_k i_o+1/4 \text{ for } k=1,2,3,\ldots,44$$

Remove excess 1.0000000000 from both arguments if any.

for $k=k_o$

transfer the 5 leftmost digits of

$$\sigma_{k_o} i_o \quad \text{and} \quad \sigma_{k_o} i_o+1/4$$

to a buffer area.

Round off both of them by adding .00005 to each.

Look up cos $\sigma_{k_o} i_o$ and cos $\sigma_{k_o} i_o+1/2$ in the table

Calculate $(y_{i_o}+y_{-i_o})\cos \sigma_{k_o} i_o$. Add the result to C_{k_o}

Calculate $(y_{i_o}-y_{-i_o})\cos(\sigma_{k_o} i_o + 1/4)$. Change the sign. Add the result to S_{k_o}

(This involves multiplying the eight individual fields of $\pm i_o$ by the <u>same</u> cos $\sigma_{k_o} i_o$ or sin $\sigma_{k_o} i_o$)

Do this for k_o= 1,2,3,....,44

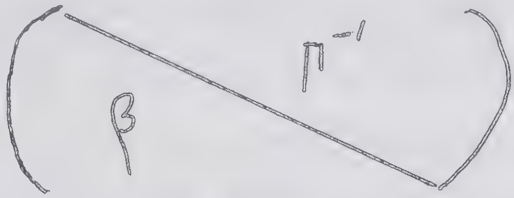
Do this for i_o = 1,2,3,....,3344

When this is done one has computed $y_o + \sum_{i=1}^{3344} (y_i + y_{-i}) \cos \sigma_k i = c_k$ and $\sum_{i=1}^{3344} (y_i - y_{-i}) \sin \sigma_k i = S_k$ for $k = 1,2,3,\ldots,44$.

One needs to multiply these C and S vectors by some special matrices which we will call the A and B matrices.

The Matrix Multiplication

The A matrix has dimension 45x45; the B matrix has dimension 44x44. They are on punched cards, in fortran format, in floating point and they are spread row wise. These matrices really consist of the juxtaposition of two triangular matrices



The lower one, the β matrix, has 0 diagonal elements while the upper one, the Γ^{-1} matrix has non-zero diagonal elements, so that their juxtaposition forms a perfect square matrix. We note that the first row of the A and B matrices is made up wholly of elements from the Γ^{-1} matrix and so on. The Γ^{-1} matrix will operate on the C (or S) vectors while the β matrix operates on the X(or Y) vectors. For example, multiplying by the A matrix can be seen as performing the following operation:

$$\begin{pmatrix} \Gamma^{-1}_{oo} & \Gamma^{-1}_{o1} & \cdots & \Gamma^{-1}_{o,44} \\ \beta_{o1} & \Gamma^{-1}_{11} & \cdots & \Gamma^{-1}_{1,44} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{o,44} & \beta_{1,44} & \beta_{43,43} & \Gamma^{-1}_{44,44} \end{pmatrix} \times \begin{pmatrix} C_o & X_o & \cdots & X_o \\ C_1 & C_1 & \cdots & X_1 \\ \vdots & \vdots & \vdots & \vdots \\ & & & X_{43} \\ C_{44} & C_{44} & \cdot & C_{44} \end{pmatrix} = \begin{pmatrix} X_o \\ X_1 \\ \vdots \\ X_{43} \\ X_{44} \end{pmatrix}$$

Inside the computer it is evidently more convenient to slap the X's on top of the C's to effect the multiplication by the next row of the matrix.

Using a matrix such as the A and B matrix on the C and S vectors is equivalent to multiplying these vectors by the inverse of the matrices of the coefficients of the unknown vectors X and Y.

Having understood this, one proceeds as follows:

First clear some buffer area. There one wants to accumulate the sum of the $A_{ik}C_k$ or $B_{ik}S_k$.

Take the first elements of the first row of the A matrix (i.e. A_{oo}) and multiply it by C_o. This means: multiply <u>each</u> of the eight fields making up C_o by the same A_{oo}.

Add this to the buffer area.

Then move to A_{o1} and calculate $A_{o1}C_1$ and add to the buffer area and so on till $A_{o,44}C_{44}$ is calculated and added to the buffer area. Then one has obtained

$$\sum_{k=0}^{44} A_{ok}C_k = X_o \text{ in the buffer area.}$$

Transfer X_o on top of C_o (wipe it out)

Then compute $\sum_{k=0}^{44} A_{1k}C_k = X_1$ and put it on top of C_1 and so on till X_{44} is calculated.

In a similar fashion go on, using now the B matrix, to calculate first

$$\sum_{k=1}^{44} B_{1k}S_k = Y_1$$

and so on, wiping out the S_i's by the Y_i's till Y_{44} is calculated.

Then in the vector area one has

$$X_o, X_1, \ldots, X_{44}, Y_1, Y_2, \ldots, Y_{44}$$

The Amplitude and Phase

Using the computer subroutines one evaluates

$$A_i = (X_i^2 + Y_i^2)^{\frac{1}{2}}$$

$$a_i = \text{arc tan } (Y_i/X_i) \quad i=1,2,3,\ldots,44.$$

A good place to store this material is in the output area since this is nearly the answer. One only needs to modify the a_i's by the astronomical arguments V_i to get the Greenwich phase lag. One should also think of transferring X_o into the output area. X_o is identical to A_o. From now on it will be called A_o.

The Calculation of V_i, the Astronomical Argument

At this stage, one reads into the computer a deck of 8 cards containing the following information:

1) 2 cards containing each 10 fields 7 digits wide which is the following matrix written out row-wise:

$$\begin{pmatrix} s_o & \frac{\Delta s}{\Delta Y} & \frac{\Delta s}{\Delta Y_1} & \frac{\Delta s}{\Delta Y_2} & \frac{\Delta s}{\Delta Y_3} \\ h_o & \frac{\Delta h}{\Delta Y} & \frac{\Delta h}{\Delta Y_1} & \frac{\Delta h}{\Delta Y_2} & \frac{\Delta h}{\Delta Y_3} \\ p_o & \frac{\Delta p}{\Delta Y} & \frac{\Delta p}{\Delta Y_1} & \frac{\Delta p}{\Delta Y_2} & \frac{\Delta p}{\Delta Y_3} \\ N'_o & \frac{\Delta N'}{\Delta Y} & \frac{\Delta N'}{\Delta Y_1} & \frac{\Delta N'}{\Delta Y_2} & \frac{\Delta N'}{\Delta Y_3} \end{pmatrix} \qquad (1)$$

2) 1 card giving the number of complete days elapsed at the beginning of each month of the year. This is written out so

0100002031030590409005120061510718108212092431027311304123334

The first two digits of the 5 digit field give the month number, followed by the number of days elapsed at the beginning of this month. Thus for example

11304: for November, 304 days elapsed (for a non leap year). The above table assumes a 365 day year.

3) 5 cards giving 44 sets of 4 integers, positive or negative

$$i_1 i_2 i_3 i_4$$

in the format xxxxxxxx, one space for the negative sign if there, one space for the digit etc.

These sets of integers are characteristic of each constituents.

The phase V_i of the ith constituent at the instant of lunar transit above or below Greenwich is given by

$$V_i=i_1s+i_2h+i_3p+i_4N'$$

where

s = the mean longitude of the moon
h = the mean longitude of the sun
p = the longitude of the moon's perigee
N'= -N, N being the longitude of the ascending node of the moon

These angles can be known at any time provided we know their value at a given instant of time and their rate of change per unit time. Since time is measured in the mixed units of years, months, days, hours and minutes, we have to know the rate of change of these angles in terms of these variables. Hence the need for the matrix (1) which gives first the value of these angles at some time origin, s_o, h_o, p_o, N'_o. In our case we choose the time origin to be

00 hour oo minute 01 January 1940

We exclude the month as a time variable because it is of unequal length and of no use really in measuring time. The other elements in the matrix give the rate of change of the angles per year (Y), day (Y_1), hours (Y_2) and minutes (Y_3). By year, we always mean a 365 day year.

It is best to evaluate first s,h,p,N' at the two desired instants of time. Afterwards, using the $i_2i_3i_4i$, one evaluates the individual V_i's.

In order to calculate s,h,p,N', we must know

$$YY_1Y_2Y_3$$

the number of 365 day years, days, hours and minutes elapsed since the time origin at the instant of upper and lower transit. These latter instants are given by the two fields of the form

$$ddxmmyyt_ot_ot_ct_o$$

in the output area.

The number of years elapsed is given by

$$Y = yy-40$$

The number of days elapsed within year yy is found by looking for the corresponding mm in the table giving the number of days elapsed at the beginning of month mm. One adds this number of days to dd. Let us call this number

$$dd'$$

Really dd' should be reduced by 1 since when one says for instance, December 13th, really only 12 days are yet completed in December. But since 1940 is a leap year, this creates an extra day that has been intercalculated since the time origin. So one should leave dd' alone.

But since a certain number of years that have elapsed since the time origin are leap years, one must reckon how many extra days have to be added to dd' in order to keep the measure of time straight.

To do this, let us divide Y by 4 in a buffer area:

1) If the remainder is different from 0, yy is not a leap year. Add the quotient q resulting from the division to dd'. Then

$$Y = dd' + q$$

the number of whole days elapsed since the time origin in excess of a whole number of 365 day years.

2) If the remainder is equal to 0, yy is a leap year. If the month is March or later, mm   03 one adds q to dd'

$$y = dd' + q$$

If the month is earlier than March, mm   03, then

$$Y_1 = dd' + q -1$$

Naturally

$$Y_2 = \text{the hour part of } t_o t_o t_o t_o$$

$$Y_3 = \text{the minute part of } t_o t_o t_o t_o$$

At this stage one has obtained the two fields of the form

$$Y\ Y_1 Y_2 Y_3$$

corresponding to the times of upper and lower lunar transit.

One then computes

$$s = s_o + Y\frac{\Delta s}{\Delta Y} + Y_1\frac{\Delta s}{\Delta Y_1} + Y_2\frac{\Delta s}{\Delta Y_2} + Y_3\frac{\Delta s}{\Delta Y_3}, \text{ etc.}$$

One is then in a position to calculate

$$V_i = i_1 s + i_2 h + i_3 p + i_4 N' \quad \text{for } i = 1,2,3,\ldots,44.$$

Matrix (1) is given in fractions of 2π; therefore the V_i's are expressed in fractions of 2π as well. One must then reduce excess 1.00 found in them. Once this is done, it is best to translate the V_i's into radians, since the a_i's obtained through a computer subroutine must certainly be in radians.

Once the two kinds of angles are compatible, one calculates

$$g_i = V_i + a_i \quad i=1,2,3,\ldots,44.$$

It might happen that some of the g_i's are negative. Add 2π radians to those that are negative. Finally translate all the g_i's into degrees.

The analysis is then over.

Output Format

For code C=1 and 4 one should insert in the output below the times of transit

Upper Transit				Lower Transit			
Low Water		High Water		Low Water		High Water	
Time	Height	Time	Height	Time	Height	Time	Height

For Code C=2 and 3, write

Lower Transit				Upper Transit			
Low Water		High Water		Low Water		High Water	
Time	Height	Time	Height	Time	Height	Time	Height

For Code C=5 and 8

	Lower Transit				Upper Transit			
	High Water		Low Water		High Water		Low Water	
	Time	Height	Time	Height	Time	Height	Time	Height

For Code C=6 and 7 Write

	Upper Transit				Lower Transit			
	High Water		Low Water		High Water		Low Water	
	Time	Height	Time	Height	Time	Height	Time	Height

and then output below this heading

	Min. °	Feet °	Min. °	Feet °	Min. °	Feet °	Min. °	Feet °
00	A_o	A_o	A_o	A_o	A_o	A_o	A_o	A_o
01	A_1 g_1	A_1 g_1	A_1 g_1	A_1 g_1	A_1 g_1	A_1 g_1	A_1 g_1	A_1 g_1
02								
44	A_{44} g_{44}	A_{44} g_{44}	A_{44} g_{44}	A_{44} g_{44}	A_{44} g_{44}	A_{44} g_{44}	A_{44} g_{44}	A_{44} g_{44}

Not forgetting to include the fields

TCxssss

and the instants of upper and lower transit in the output as well.

Specifications for the Analysis of 19 years of Discontinous Observations on the Height and Time of High and Low Water.

Summary

The input deck contains 19 years of observations on the time and height of low and high water carried through during the navigation season.

The analysis itself is performed on 19 sets of observations and each set consists of 177 lunar days of observations followed by a lacuna of 175 lunar days. (1)

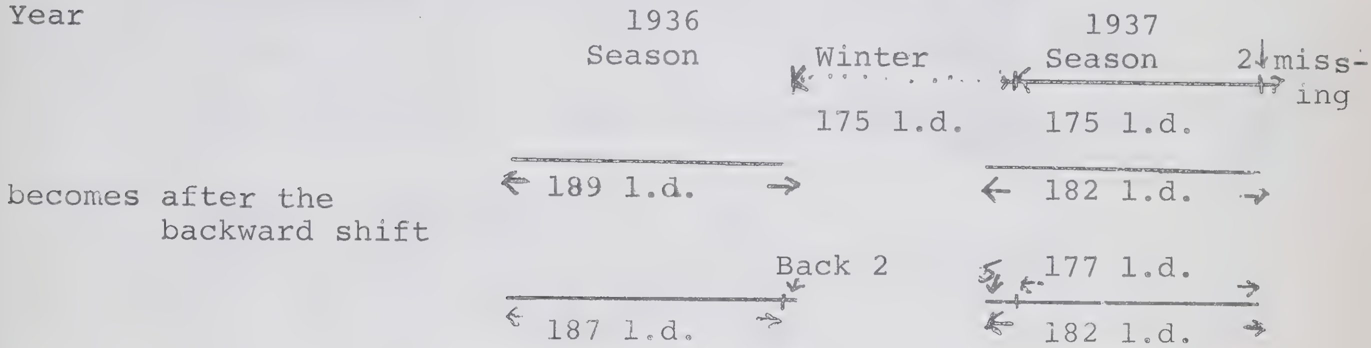
The first part of the program will consist in searching for the Gregorian date of each lunar day of observation over the interval of 19 years. A register will be built up of such Gregorian dates complete with a set of 19 counters giving the number of complete lunar days of observations available over each year. (One complete lunar day of observations consists of a sequence of 4 observations on the time and height of high and low water beyond the moment of lunar transit.)

The jump of 176 lunar days will shift earlier and earlier in the spring of the following year, the date for the first lunar day of observations. The observations themselves follow the seasons and not the lunar calendar. There is then a danger that after a few years, the date required for the first day of observations in order to fulfill (1) might fall ahead of the first actual day of observations. If this is so, the analysis is impossible and the program is terminated as soon as this is realized.

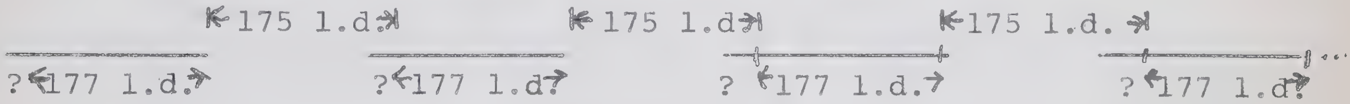
To make the best use of the available data, one will always jump 176 lunar days (182 solar days, 4 hours and 03 minutes) from the last complete day of observations of the year. As the first year is processed, the Gregorian date of each lunar day of observation is listed in the date register, so that once one has reached the last full day of observations the counter shows that there are more than 177 lunar days of observations for this year. One then jumps 176 lunar days and usually lands within the series of existing observations for the following year. One notes the Gregorian date of the first lunar day of observations, then moves 177 lunar days later. This might lead beyond the point where observations exist for this year. To correct this, one moves back one or more lunar days within all the observations collected up to that point, till the 177th day falls on a day with observations available.

In the course of this retreat in time, the latter lunar days of observations of the previous year are erased from the Gregorian date register. If any of the counters becomes less than 177 in this process, the program is terminated. If this does not occur, one proceeds to read the following year of data, following the same procedure, till the 19 years of observations are read in. The Gregorian date calendar at this stage list the datas of all the lunar days in each year which allow the satisfaction of criterion (1) from the right. The 19 counters should each contain a number equal to or larger than 177.

One may illustrate the initial steps of the elimination from the right of the excess data as one jumps 176 days and tries to find a set of data which satisfies criterion (1).



The solid line indicates the time of the year during which observations are available. At the end of this loop one obtains the following arrangement of Gregorian dates:



Now one is ready to accumulate the data which will be used for the analysis. Going back on the first input card which is kept somewhere on a tape, one looks up the counter for the first year of observations. In the Gregorian date register one jumps the first few dates if the counter is in excess of 177, and jumps as many as the difference is between the counter and 177.

Then one looks for the card for which the first acceptable Gregorian date corresponds and one starts building 177 data out of this year. One goes then to the next set, jumps the first few superfluous Gregorian dates (if any), and by looking up the input cards goes on filling continuously the data area, without any break between the data for one year and next.

From this point on the analysis is pretty well similar to that of 19 years of continuous observations with two important exceptions:

1) There are two sets of σ's in the σ cards. One set gives σ in fractions of 2π /lunar day as before

The other gives

$\Delta\sigma$ in fractions of 2π which is the change in the phase of each constituent over an interval of 175 lunar days.

2) When the column vectors C and S are computed, one increments

$$\sigma_k i \quad \text{and} \quad \sigma_k i + 1/4$$

by σ_k for 177 consecutive cycles. Then one adds

$$\Delta\sigma_k$$

to

$$\sigma_k i \qquad \text{and} \qquad \sigma_k i + 1/4$$

before going on to the next set of 177 data.

The A and B matrices are also different in content from those pertaining to the analysis of continuous data and they are 44 x 44 and 43 x 43, but the mechanical steps for the calculations of the X and Y vectors, are identical to those described in the previous program.

Note: To understand what follows one must be familiar with the specifications I gave for the analysis of 19 years of <u>continuous</u> observations on the time and height of high and low water.

DETAILS

Preliminaries

The input card format is identical to the one used for the analysis of continuous records.

As the first part of the program is performed, the cards scrutinized should be stored sequentially on an auxiliary tape because one will have to read them again once the Gregorian date register is established.

Read the first card.

Note the code C

if C =	(1 or 4		(1		(LU
	(2 or 3		(2		(LL'
	(5 or 8	Insert in core Memory	(3	which means	(HL'
	(6 or 7		(4		(HU

where L = Low water
H = High water

L'= Lower lunar transit
U = Upper transit

This keeps note of the order in which the high and low water are organized with respect to the lunar transit on the first lunar day of observations.

The Gregorian Date Register

This register will contain a sequence of Gregorian (calendar) dates of the form

ddxmmyyxtttt

dd = day number
mm = month number
yy = last two digit of the year number
tttt = hours tttt and minutes tttt
x = blank

The above field indicates the time of the first observation on a given lunar day. On any lunar day, there will be four

observations on high and low water, each one being of the form

xttttxhh.h (1)

giving the clock time and the height to the tenth of a foot of the extremum. In building the Gregorian date register, one therefore looks up only the first, 5th, 9th, etc., observations in the sequence of data.

One sets up a special register for each year. For any register, except for the first year, one must allow some space to the left of it, since an individual register is not filled from the left, but rather outwards to the right and the left from a given point.

To get the first data while the first input card is in the reader, one transfers the field

ddxmmyy (2)

which is in column 8 to 15 of the card, to the register for the first year. Then if the code

C = 1, 3, 6, 8

one adjoins to (2) the field xtttt pertaining to the <u>first</u> observation on the card. If

C = 2, 4, 5, 7

one adjoins the xtttt pertaining to the second observation on the card. One has then formed in the date register the field

ddxmmyyxtttt (3)

which is the Gregorian date of the first observation on the first lunar day of the first year.

One goes on reading another card, forming in sequence the Gregorian date for the 5th, 9th, etc., (or 6th, 10th,etc.) and storing them in the date register for the first year, keeping note as well in a counter of the number of Gregorian dates being accumulated.

One does not count as an observation any observation encoded as

x9999 99.9 (4)

which indicates that an extremum is missing over a given solar day. One simply ignores fields of the form (4).

One proceeds in this fashion till the register for the first year is completed. This is noted by a change in yy.

One should then check carefully if the last Gregorian date registered corresponds to a complete sequence of 4 observations of the form

xtttxhh.h

A.- As soon as a change in yy is noted, one is in presence of the first observation for the following year and the first card of observations for this year is in the reader.

One must now perform the jump of 176 lunar days (182 solar days 4 hours and 03 min.) The number in the counter pertaining to the register of the year that has just been processed should be equal or larger than 177. If not, send out an error message, output routine 1 (see below) and terminate the program.

To perform the jump, one transfers the last Gregorian date which has just been abstracted for the preceeding year, to some buffer area. There it is transformed into the field

dddmmyyxtttt (5)

by consulting the following table which has been stored somewhere in the computer.

	mm	number of days elapsed at the beginning of the month
January	01	00
February	02	31
March	03	59
April	04	90
May	05	120
June	06	151
July	07	181
August	08	212
September	09	243
October	10	273
November	11	304
December	12	334

One looks up mm in the table, gets the corresponding number of days and adds it to dd in order to obtain ddd in (5).

For example 1011   304 + 10 = 314   ddd
(Nov. 10th)

After this one adds 182 to ddd

and 0403 (4 hours 03 minutes) to tttt. If in excess of 60 mins. add 01 to tttt and subtract 60 from tttt.

In the reader there is still the first card of the new year of observation. So one checks if yy is a leap year (divide yy by four for instance in some buffer area).

If yy is a leap year,

subtract 1 from ddd

If yy is not a leap year,

leave ddd alone.

Now subtract 365 from ddd (always).

Using Table 1 again, transform field (5) back into the form

ddxmmyyxtttt (6)

Out of field (6), one forms two new fields by subtracting or adding 0200 (2 hours) to tttt in 6. One obtains the two new fields

ddxmmyyxtttt and dd'xmm'xt't't't' (7)

(a) (b)

which will be called (7a) and (7b). (7a) and (7b) give the limiting times between which the first observations on the day which occurs 176 lunar days after the last day of observation in the previous year, should fall.

There is a possibility that tttt and/or t't't't' be equal or larger than 2400. If this is so one subtracts 2400 and adds 01 to dd and/or d d'.

There arises then the possibility that dd or dd' exceeds the number of days allowed for a given month in 7 (for instance dd = 31 for mm = 04 April). If this is so one must consult the following table.

mm	Maximum number of days
01	31
02	28
03	31
04	30
05	31
06	30
07	31
08	31
09	30
10	31
11	30
12	31

If dd and/or dd' exceed the maximum number of days allowed for a given month as it may be checked by consulting the above table with mm and/or mm', subtract the number of maximum days from dd and/or mm' and then add 01 to mm and/or mm'.

Now one faces the task of finding within the current year some observation whose time field falls between 7a and 7b. One must first check if the observations extend early enough in the spring to allow such an observation to exist.

Recalling that the first card for the year is still in the reader and that the first time in it gives the time of the earliest observation, one first compares the mm on this card with the mm' of 7b.

If the mm on the card is larger, one sends out an error message, outputs routine 1 and terminates the program.

If the mm on the card is equal to mm', one compares the dd on the card with the dd' of 7b.

If the dd on the card is larger, one sends an error message, outputs routine 1 and terminates the program.

If the mm and the dd on the card equals the mm' and the dd' of 7b, one compares the <u>first</u> xtttt on the card with the xt't't't' of 7b.

If the first xtttt on the card is larger, one sends an error message, outputs routine 1 and terminates the program.

If all of the above tests fail, one may proceed to look for the observation which falls between time 7a and 7b, by starting to read the input cards for the year under investigation.

a) If the mm of 7a is different from the mm' of 7b.

One searches through the input deck to find the last card of observation pertaining to month mm and the first card of observation pertaining to month mm and the first card of observation pertaining to month mm' (they are consecutive).

The observation we search for is either the last observation of the last month or the first observation of the next month. (By last observation of the month one must note that we mean the last existing observation over the solar day; if the fourth field on the card for instance is x9999xhh.h, the last observation is the third field).

1) Test if the xtttt of the last observation is larger than the xtttt of 7a. If so it is the observation we are looking for. If not so

2) Test if the xtttt of the first observation of the next month is smaller than the xtttt of 7b. If it is so, this is the observation we are looking for.

If both tests fail, send an error message output routine 1 and terminate the program.

b) If the mm of 7a is equal to the mm' of 7b but if the dd of 7a is different from the dd' of 7b

Search through the deck for the cards corresponding to month mm. Then within this month search for the two consecutive cards whose dd correspond to the dd of 7a and 7b. The last existing observations on the first card or the first observation of the next card is the one we are looking for.

Do test 1) and 2) as above.

If both fail, send out an error message, output routine 1 and terminate the program.

c) the ddxmm of 7a and 7b are equal.

Then one looks through the deck to find the card whose ddxmm corresponds to the one of 7a and 7b.

One of the four fields on the cards is the one we are looking for.

Test if the xtttt of the first observation card is larger than the xtttt of 7a. If it is so, test if its xtttt is smaller than the xtttt of 7b. If it is so, it is the observation we are looking for.

If the first field has an xtttt smaller than the xtttt of 7a text, test the second field and so on.

If the sequence of the four tests fail, send out an error message, output routine 1 and terminate the program.

So finally, there is a card in the reader over which lies the observation we are looking for.

One checks first the C of the card on which it is found. This first observation should be of the same character as the first observation that was abstracted for the first observation of the first lunar day of the first year.

Then one checks the position of the observation on the card; it should be the first, second, third or fourth field, then looks up the table below which gives the following information:

Code	Order of the field on the card			
	1	2	3	4
1	1	4	2	3
2	4	2	3	1
3	3	1	4	2
4	1	3	2	4
5	1	3	2	4
6	4	1	3	2
7	2	4	1	3
8	3	2	4	1

Knowing the position of the field on the card (1,2,3, or 4) and the code C on the card, one looks up the corresponding number in the above table. This number should be identical to the number stored in the core memory at the beginning of the program. If it is not so, send an error message, output routine 1 and terminate the program.

If all is well, one is ready to start building the register for the year under scrutiny, but the way to do it is different from the way it was for the first year. What follows describes the procedure.

From the observation we have just found, which lies 176 lunar days beyond the last day of observation of the previous year, we build the Gregorian Date

ddxmmyyxtttt (8)

from the ddxmmyy on the card and the xtttt pertaining to the observation. This date is stored in the date register pertaining to the year under scrutiny, and then one proceeds to work backward.

Going back over the tape which contains all the input cards read up to now, one picks the Gregorian date, in the procedure outlined previously, from the 5th, 9th, etc., data to the left of the observation which has been used to establish (8). One proceeds so till the beginning of the year under scrutiny is sensed, storing the Gregorian dates to the left of (8). In the meantime a counter has been keeping track of the Gregorian dates entered into the register (not including in the count (8). This counter is set aside preciously.

Now one moves back over the input cards, without doing anything, till the card from which field (8) has been extracted is sensed. Then one goes on reading the cards from the reader, abstracting the Gregorian dates and storing them to the right of (8) keeping now a separate count of these dates, including (8) in the count and checking if the last date corresponds to a complete lunar day of observations.

This last counter should be larger, equal or smaller than 177. If the counter is larger or equal to 177, one adds 177 to the first counter keeping track of the dates to the left of (8). This counter is the counter which gives the number of useful data in the year just processed. If the count is smaller than 177, one computes the difference between this count and 177 (as a positive number) and subtracts this number from all the counters for the previous years processed up to then giving the

number of useful dates in those years. If any of these counters, after the subtraction is felt to be smaller than 177, an error message is sent, routine 1 is outputed and the program is terminated. For the year just processed one adds the content of the two separate counters which had been keeping track of the number of dates stored to the left and right of (8) for the given year. If this number is sensed to be less than 177, an error message is sent, routine 1 is outputed and the program is terminated.

One notes that as a consequence of this procedure, one saves all the dates in the last year processed, unless there are more than 176 dates to the right of (8). In the latter case, the extra dates to the right are not counted.

One then goes back to A.-page67 till the 19 years are edited in the way just described.

The Data Build Up

At this stage one has 19 sets of 177 or more Gregorian dates. These dates will be used to build 19 sets of 177 data which will be submitted to the analysis.

One rewinds the tape so that one may contemplate the first card of the first year of observations; one gets ready as well to look up the date register pertaining to this year as well as its counter.

B.- If this counter is larger than 177, let us say, it equals 177+x, one jumps the first x fields of the date register and picks the x+1st one.

If the counter equals 177, one picks the first date of the register. With the help of this date, one starts searching the input cards for the one containing the observation xttttxhh.h pertaining to this date. One stores in the data area the field

xDDDxhh.h

corresponding to this date. Following this field one stores the three fields

xDDDxhh.h

which follow it immediately. The sequence of four fields

xDDDxhh.hxDDDxhh.hxDDDxhh.hxDDDxhh.h

form one data pertaining to one lunar day. Then one proceeds to go continuously through the cards of this year to build the remaining 176 data.

Once the whole year has been processed one goes back to B.-page 73 till the 19 years have been processed and one has accumulated the 19 required sets of 177 data.

The Analysis

The steps for the analysis are identical to those described for the analysis of continuous data.

The only difference in the input data is the σ cards which contains two sets of angles.

Also, in the continuous case, one analyzes for 45 constituents (i=0,1,2,......,44) while in the discontinuous case one analyzes for 44 constituents (i=0,1,2,3,.....,43).

The matrices as a consequence are of smaller dimension, the A and B matrices being now 44x44 and 43x43.

So whenever the number 44 is met in the previous set of notes it should be understood as being replaced by 43 for the present analysis.

In the previous case also, the calculation of the C and S vectors was done in 3343 repetitive cycles.

Now, one goes through 177 cycles in the same fashion as described previously, but before one goes into the next cycle, one increments the arguments

$$\sigma_k i \quad \text{and} \quad \sigma_k i+1/4$$

by $\Delta\sigma_k$ which accounts for the 175 missing data. Then one goes on with the next set of 177 data till all 19 sets have been processed.

The evaluation of the X and Y vector from the C and S vectors and the A and B matrices is absolutely identical to the previous procedure described previously, with the exception that the summation are carried for up to k=43 rather than k=44 as before.

The Output

Before outputing the results of the analysis in the format recommended in the notes for the analysis of continuous data, it is best to output the first and last Gregorian date used for the analysis of each year that is to say the Gregorian date of the first and 177th data used in the analysis for a given year. Thus

Year	Date of First Obse.Used	Date of Last Obse.Used
00	ddxmmyyxtttt	ddxmmyyxtttt
01	ddxmmyyxtttt	ddxmmyyxtttt
⋮	⋮	⋮
19	ddxmmyyxtttt	ddxmmyyxtttt

Then on a separate sheet one can start outputting the standard results with the heading

CONSTITUENTS OBTAINED FROM THE ANALYSIS
OF 19 YEARS OF DISCONTINUOUS OBSER-
VATIONS ON THE TIME AND HEIGHT OF HIGH WATER.

TC sssss etc.

Routine 1

This routine should output:

1) TCxssss, the code and station number.

2) The content of the pair of counters used in building up the date registers.

   (One must recall that for the first year, there is only need for one counter).

3) A message describing the type of error.

4) A list of all the Gregorian dates accumulated in each register at the moment the program was terminated.

Specifications for the Smoothing and the Low Passing of the Data on High and Low Water.

Summary

The smoothing and the low passing operations have to be performed in such a way that the output may be used without ambiguity in the program for the analysis of 19 years of observations. Since the first filtered field can fall anywhere on a given card, it is not practical to start outputting with this very field and it is preferable to output to original card containing some unfiltered field if necessary.

The smooting operator chosen is $a_2^2\, a_3$ while the low pass filter is $a_{26}\, a_{28}\, a_{31}$

Card Format

It is described in our specifications for the analysis of 19 years of observations on the time and height of high and low water.

Smoothing

One needs five fields to apply $a_2^2\, a_3$.

Initially 5 or more cards are read in order to accumulate in storage 5 fields of the form

```
xDDDxhh.hxDDDxhh.hxDDDxhh.hxDDDxhh.h
 1    2    3    4    5   6   7    8
```

Somewhere else as well the images of the cards used must be kept (at least the one(s) containing the third unsmoothed field).

One applies $a_2^2\, a_3$ to the 5 fields (8 identical operations) and one obtains a smoothed field falling over the third field. The position of the xDDD field of the third field on the original data card is looked up and the smoothed field is applied over the original data, one card being outputed each time it is full. This implies that the first card outputed might contain some unsmoothed fields to the left of xDDD. Then one goes on reading more cards and smoothing till the data deck is exhausted.

Low Passing

One needs 83 fields to apply $\mathcal{a}_{26}\mathcal{a}_{28}\mathcal{a}_{31}$. One uses the smoothed fields to apply this operator.

Initially one reads as many cards as needed to obtain 83 fields in storage and one must retain the image of the card on which the first portions of the 42nd field falls as well as those that follow it.

One applies the operator to the 83 fields; the low pass field falls over field no. 42. One checks in the card image memory over which portion of the card the low pass xDDD lies. If it is not the first on the card, one repeats the low pass field to the left of itself and one moves backwards from xDDD till the portion of the field is reached which corresponds to the first field on the card. When this position is reached one starts computed y smoothed -y low pass and outputs a card identical in format to card no. 42. Then one moves one lunar day and repeats the same operation till the end of the data deck is reached.

Somehow one must also output y smoothed in a card format identical to the original data.

Specifications for the Verification of the Analysis on 19 years of Continuous Observations on the High and Low Waters and for Predictions.

Summary

On the first card, the first and third tttt and DDDD fields are used to evaluate the time of lower and upper transit.

s,h,p,N' are computed for these two instants. Then the i's are used to evaluate V_i for i=1,2,3,....,44.

With the help of the analyzed g_i's, the 8 sets of V_i-g_i are evaluated. Then one calculates

$$A_o + \sum_{i-1}^{44} A_i \cos(V_i-g_i) \text{ for the eight sets; this}$$

gives the predicted DDDxhh.hxDDDxhh.hxDDDxhh.hxDDDxhh.hx

for the first lunar day. Compute the difference between the observed and predicted fields. Plot the results.

Add σ_i to the set of 8 V_i-g_i; this gives the phase for the second lunar day.

Compute

$$A_o + \sum_{i=1}^{44} A_i \cos(V_i-g_i+\sigma_i);$$ this gives the

predicted field for the second lunar day. Compute the difference between observed and predicted. Plot.

Add σ_i to the argument to get to the third day etc. and proceed so, till the last day of December 1964 is reached.

Calculate the predicted field for the period 1965-1971 (excluded), output the results in a format suitable for reuse in the computer.

Details:

With ddmmyy and the tttt and DDD pertaining to the first and third field on the first card, we get the date and time of the lower and upper transit corresponding to the first set of data. Compute the numbers of years and fractions of years elapsed between 00hr. 00min. 01 January 1940 and these two transits. Call these differences Δ_1 and Δ_2.

Compute

$$s=s_o+\Delta_1\frac{\Delta s}{\Delta year},\ h=h_o+\Delta_1\frac{\Delta h}{\Delta year}$$ etc. for these two

transits.

With the i's compute the two sets of 44 V's given by

$$V_{1i}=i_1s_1+i_2h_1+i_3p_1+i_4N'_1 \text{ and } V_{2i}=i_1s_2+i_2h_2+i_3p_2+i_4N'_2$$

With the help of the analyzed g_i's compute the eight sets

of $$V_i-g_i$$

V_{1i} pertaining to the first set of four; V_{2i} pertaining to the second set.

V_i-g_i gives the phase of the 44 constituents on the first lunar day.

Compute the sum

$$A_o+\sum_{i=1}^{44} A_j\cos(V_i-g_i)$$

with the help of the eight sets of analyzed A_i's. This gives

the predicted field

DDDxhh.hxDDDxhh.hxDDDxhh.hxDDDxhh.hx

for the first day. Compute the difference between the observed and predicted field. Plot the observed field, the predicted field and the residue with separate plots for DDD,hh.h,DDD,hh.h, DDD,hh.h,DDD,hh.h with proper labelling.

Add σ_i to the eight sets of V_i-g_i; this gives the phase of the constituents on the second lunar day. Compute

$$A_o+\sum_{i=1}^{44} A_i\cos\left(V_i-g_i+\sigma_i\right);$$ this gives the

predicted field for the second lunar day. Compute the difference between the observed and the predicted and plot the three quantities.

Proceed along these lines till the card pertaining to 31 Dec. 1964 is sensed.

Do not compute the residues, nor compare with the observed quantities from this point on and compute the predicted values for the interval 1965-1971 (excluded) separately from the previous work and outputed in a format suitable for re-use in the computer.

Appendice 1

Liste des fréquences repliées suggérées par l'institut hydrographique allemand.

k	σ_k (cycles/jour lunaire)
0	.0000000000
1	.0001522497
2	.0028338733
3	.0050271437
4	.0056677466
5	.0322162254
6	.0325375295
7	.0374114225
8	.0375636722
9	.0377317239
10	.0378839737
11	.0380362234
12	.0432314188
13	.0650730570
14	.0672663275
15	.0699479510
16	.0701002007
17	.0702524505
18	.0704205021
19	.0751273445
20	.0754476459
21	.0756156976
22	.0757679473
23	.0759200304
24	.1026367292
25	.1048299997
26	.1075116232
27	.1076638729
28	.1078161227
29	.1079841744
30	.1083044758
31	.1126910167
32	.1133316195
33	.1373665282
34	.1402004014
35	.1452275452
36	.1458681480
37	.1515358946
38	.1727369300
39	.1777640737
40	.1780843751
41	.1834318202
42	.2103006022
43	.2159683487
44	.2804008030

Appendice 2

0.14962251E -3	0.52513922E -5	-.74389186E -6	0.25207847E -5	4
0.54150605E -6	0.41467800E -6	0.37450192E -6	-.21220517E -6	8
0.20497789E -6	-.30354542E -6	0.32122886E -6	-.33030512E -6	12
0.17616915E -6	0.15942624E -6	0.34784284E -7	0.72782093E -7	16
-.62456581E -7	0.53047687E -7	0.37515053E -7	-.16406120E -6	20
-.13198572E -6	0.82738999E -7	-.81837768E -7	0.76641007E -7	24
-.14078482E -6	0.84095947E -7	0.48059673E -7	-.52170578E -7	28
0.52900249E -7	-.89484196E -7	-.11374573E -6	0.74749435E -7	32
-.36817937E -7	-.48989335E -7	0.60568140E -7	0.92989465E -7	36
0.72225965E -7	0.88050435E -7	0.83734330E -7	0.14783527E -7	40
0.49959579E -7	-.48086946E -8	-.57284230E -7	-.65283735E -7	44
0.53799203E -7	0.35097608E -1	0.29381843E -3	0.15472838E -5	48
-.51305700E -5	-.11084812E -5	-.84261776E -6	-.76108672E -6	52
0.43108138E -6	-.41640643E -6	0.61676355E -6	-.65269949E -6	56
0.67114969E -6	-.35785995E -6	-.32390927E -6	-.70735937E -7	60
-.14793581E -6	0.12695476E -6	-.10783493E -6	-.76189150E -7	64
0.33336649E -6	0.26820755E -6	-.16815583E -6	0.16632650E -6	68
-.15576814E -6	0.28607261E -6	-.17086275E -6	-.97637803E -7	72
0.10599170E -6	-.10747545E -6	0.18182098E -6	0.23112827E -6	76
-.15190342E -6	0.74795028E -7	0.99556223E -7	-.12308260E -6	80
-.18894887E -6	-.14676819E -6	-.17891978E -6	-.17014391E -6	84
-.30032746E -7	-.10151077E -6	0.97780910E -8	0.11640276E -6	88
0.13265613E -6	-.10931903E -6	-.51566194E -2	0.52661225E -2	92
0.29988767E -3	-.73246904E -5	-.10013698E -5	-.85792749E -6	96
-.76942931E -6	0.44371901E -6	-.42824826E -6	0.62821223E -6	100
-.66449987E -6	0.68283295E -6	-.35857537E -6	-.32968596E -6	104
-.68771375E -7	-.14736501E -6	0.12614186E -6	-.10686652E -6	108
-.79073380E -7	0.33710472E -6	0.27034065E -6	-.16837178E -6	112
0.16642494E -6	-.15568589E -6	0.28891613E -6	-.17350024E -6	116
-.99539762E -7	0.10794018E -6	-.10938580E -6	0.18409136E -6	120
0.23347308E -6	-.15270390E -6	0.76441964E -7	0.99748860E -7	124
-.12380154E -6	-.19095734E -6	-.14785604E -6	-.18048024E -6	128
-.17188772E -6	-.30673240E -7	-.10277522E -6	0.95241570E -8	132
0.11737675E -6	0.13385446E -6	-.11032893E -6	0.17334545E -1	136
-.17333079E -1	-.24424782E -1	0.30088215E -3	-.16144204E -4	140
0.29900296E -6	0.21300724E -6	-.21199224E -6	0.20109419E -6	144
-.23382324E -6	0.24439829E -6	-.24683455E -6	0.18310592E -6	148
0.13173383E -6	-.49316772E -8	0.26923633E -7	-.19732694E -7	152
0.13858833E -7	0.47149840E -7	-.11395465E -6	-.82562640E -7	156
0.40040602E -7	-.38409311E -7	0.34154340E -7	-.95758569E -7	160
0.67056177E -7	0.42475447E -7	-.44897150E -7	0.44844815E -7	164
-.65826621E -7	-.78031453E -7	0.43623169E -7	-.34569861E -7	168
-.27679048E -7	0.36342609E -7	0.65286058E -7	0.45786478E -7	172
0.58309007E -7	0.58290549E -7	0.13765344E -7	0.37130577E -7	176
0.34337981E -9	-.37686992E -7	-.43883452E -7	0.36467103E -7	180
0.46644479E -2	-.46651176E -2	-.46496915E -2	-.53656239E -1	184
0.29967630E -3	0.84295433E -6	0.74210176E -6	-.44754681E -6	188
0.43102487E -6	-.61717029E -6	0.65201551E -6	-.66886058E -6	192
0.37198006E -6	0.32268104E -6	0.59347155E -7	0.13630235E -6	196
-.11585835E -6	0.97452145E -7	0.81143592E -7	-.32442594E -6	200
-.25800308E -6	0.15789755E -6	-.15578317E -6	0.14529333E -6	204
-.27706676E -6	0.16870796E -6	0.97760917E -7	-.10572593E -6	208
0.10698133E -6	-.17768254E -6	-.22400758E -6	0.14467664E -6	212
-.75539013E -7	-.94448350E -7	0.11745829E -6	0.18341789E -6	216
0.14085106E -6	0.17250850E -6	0.16493333E -6	0.30254158E -7	220
0.99170090E -7	-.82635681E -8	-.11208215E -6	-.12803499E -6	224
0.10558466E -6	0.28270494E -2	-.28223490E -2	-.28234783E -2	228
0.11446831E -2	0.28128829E -2	0.30024202E -3	-.19195793E -4	232
-.15410530E -5	0.13630546E -5	-.40103210E -6	0.34304934E -6	236
-.24113614E -6	0.86205407E -6	0.16749278E -6	-.25826999E -6	240
-.21026849E -6	0.20900805E -6	-.20205578E -6	0.17542126E -6	244
0.17411914E -7	0.89064721E -7	-.15224447E -6	0.16027966E -6	248
-.16489194E -6	0.27282351E -7	0.57328339E -7	0.63451817E -7	252

-.59873524E -7	0.55650324E -7	-.19728941E -7	0.16744738E -7	256
-.67195235E -7	-.63106233E -7	0.49185998E -7	-.45514196E -7	260
-.20913760E -8	-.36609036E -7	-.26788402E -7	-.52066738E -8	264
0.23385444E -7	0.13360672E -7	0.25061325E -7	0.18630854E -7	268
0.14736168E -7	-.99030204E -8	0.27375855E -2	-.27333911E -2	272
-.27174369E -2	0.91399838E -3	0.26561845E -2	-.63934401E -1	276
0.29923846E -3	-.24102537E -5	0.21759461E -5	-.15112178E -5	280
0.14879710E -5	-.13884613E -5	0.12454709E -5	0.36269799E -6	284
-.21383621E -6	-.12224829E -6	0.13327512E -6	-.13755727E -6	288
0.21780641E -6	-.16963164E -6	-.61923964E -7	-.56648632E -7	292
0.65702086E -7	-.76235762E -7	-.11790273E -6	0.14270324E -6	296
0.11158172E -6	-.11222238E -6	0.10878185E -6	-.11054759E -6	300
-.99175710E -7	0.91573180E -8	-.99402053E -7	0.47578521E -9	304
0.14359397E -7	0.88544083E -7	0.34243068E -7	0.59013583E -7	308
0.75216636E -7	0.37209956E -7	0.61006995E -7	0.21093944E -7	312
-.35962497E -7	-.47310258E -7	0.40734589E -7	-.14063947E -2	316
0.14036656E -2	0.14190845E -2	-.77146385E -3	-.14576019E -2	320
-.56476723E -2	-.80546256E -2	0.30125103E -3	0.20944082E -5	324
-.18835682E -4	0.14153859E -4	-.11700215E -4	-.13025452E -6	328
0.18196187E -6	0.26107164E -6	0.29015399E -6	-.27008666E -6	332
0.24749422E -6	-.62996062E -7	-.27696761E -6	-.29344341E -6	336
0.25457545E -6	-.25869348E -6	0.25291031E -6	-.23074875E -6	340
0.85254225E -7	0.26725820E -7	-.35305286E -7	0.39323405E -7	344
-.11865787E -6	-.18002504E -6	0.15643988E -6	-.97366127E -8	348
-.10295364E -6	0.11641936E -6	0.13123803E -6	0.12599806E -6	352
0.14064488E -6	0.11807639E -6	0.41335921E -8	0.58965695E -7	356
-.24387575E -7	-.89838060E -7	-.97795276E -7	0.78168981E -7	360
0.13710867E -2	-.13684500E -2	-.13823047E -2	0.73904313E -3	364
0.14163505E -2	0.50440236E -2	0.73276109E -2	0.69523686E -2	368
0.30421300E -3	0.32290191E -4	-.18867369E -4	0.14046993E -4	372
0.18957486E -6	-.15485592E -6	-.25321462E -6	-.27753720E -6	376
0.25895911E -6	-.23778921E -6	0.66368145E -7	0.27454000E -6	380
0.27460586E -6	-.24170343E -6	0.24566504E -6	-.24074193E -6	384
0.21360212E -6	-.75926550E -7	-.21790667E -7	0.29845000E -7	388
-.33729354E -7	0.10833077E -6	0.16641893E -6	-.14692041E -6	392
0.61803694E -8	0.46555083E -7	-.10907200E -6	-.12089789E -6	396
-.11748715E -6	-.13052563E -6	-.10881646E -6	-.27917565E -8	400
-.53650902E -7	0.23551853E -7	0.83412082E -7	0.90560768E -7	404
-.72296477E -7	-.22343992E -2	0.22804208E -2	0.22862214E -2	408
-.10081624E -2	-.22837551E -2	-.25470792E -2	-.63315914E -2	412
-.63262805E -1	0.10014162E 0	0.29908283E -3	0.54039939E -5	416
-.53851396E -5	-.13706752E -5	-.74517920E -7	0.35164404E -6	420
0.31824706E -6	-.30812436E -6	0.29175253E -6	-.18610694E -6	424
-.14652095E -6	-.20850064E -6	0.24731858E -6	-.25619433E -6	428
0.25766323E -6	-.12475649E -6	-.10960921E -7	-.41801033E -7	432
0.34996167E -7	-.29656301E -7	-.35599667E -7	-.93399706E -7	436
0.12539561E -6	0.48776467E -7	-.86193140E -7	0.39026508E -7	440
0.59480001E -7	0.85790603E -7	0.84056364E -7	0.56233364E -7	444
-.17443337E -7	0.14934058E -7	-.32175426E -7	-.54809468E -7	448
-.55098092E -7	0.42468978E -7	0.23647849E -2	-.23697035E -2	452
-.23628104E -2	0.10234584E -2	0.23568993E -2	0.20846080E -2	456
0.59198167E -2	0.48557851E -1	-.63937068E -1	0.18068553E -1	460
0.29900464E -3	0.54813853E -5	0.15644556E -5	0.10939536E -6	464
-.36136593E -6	-.31951709E -6	0.31088436E -6	-.29545850E -6	468
0.20154723E -6	0.12455765E -6	0.19501670E -6	-.24418818E -6	472
0.25363963E -6	-.25609118E -6	0.10903828E -6	0.24034599E -7	476
0.50780968E -7	-.44284824E -7	0.36815484E -7	0.23880386E -7	480
0.80631639E -7	-.11992165E -6	-.56303284E -7	0.83037691E -7	484
-.84388241E -7	-.49071183E -7	-.79430350E -7	-.75472521E -7	488
-.47210516E -7	0.20213486E -7	-.87657255E -8	0.32910717E -7	492
0.49451835E -7	0.48699814E -7	-.37154943E -7	-.24388297E -2	496
0.24346563E -2	0.24350567E -2	-.10359689E -2	-.24236970E -2	500
-.16361230E -2	-.55136862E -2	-.41189018E -1	0.49257336E -1	504
-.18336747E -1	0.18332108E -1	0.29892514E -3	-.17663019E -5	508
-.14485994E -6	0.37019394E -6	0.31980951E -6	-.31264946E -6	512
0.29832446E -6	-.21673858E -6	-.10167626E -6	-.18059809E -6	516

0.24021697E -6	-.25022843E -6	0.25367858E -6	-.92653694E -7	520
-.37307005E -7	-.59790611E -7	0.53629532E -7	-.48044824E -7	524
-.11844867E -7	-.67351001E -7	0.11396655E -6	0.63817079E -7	528
-.79565486E -7	0.79400068E -7	0.38308087E -7	0.72702343E -7	532
0.66490379E -7	0.37869867E -7	-.22975067E -7	0.24522606E -8	536
-.33559602E -7	-.43850955E -7	-.42031894E -7	0.31628251E -7	540
0.11392686E -2	-.11369984E -2	-.11502978E -2	0.65118417E -3	544
0.11835091E -2	0.30994677E -2	0.40615166E -2	-.12240844E -2	548
0.17351802E -2	-.47858157E -2	0.53405335E -2	-.59088439E -2	552
0.29885639E -3	-.30559479E -6	-.35053178E -6	-.39476519E -6	556
0.36507001E -6	-.33260987E -6	0.65401890E -7	0.42311751E -6	560
0.40682501E -6	-.34154933E -6	0.34602954E -6	-.33686914E -6	564
0.30637993E -6	-.12195239E -6	-.44189513E -7	0.55167530E -7	568
-.59962470E -7	0.16112636E -6	0.23815177E -6	-.19913537E -6	572
0.21295591E -7	0.12848889E -6	-.14642179E -6	-.17108047E -6	576
-.15976293E -6	-.17980836E -6	-.15235320E -6	-.83182869E -8	580
-.78029095E -7	0.28240530E -7	0.11342498E -6	0.12409874E -6	584
-.99012013E -7	0.10811578E -2	-.10792204E -2	-.10789418E -2	588
0.49424665E -3	0.10722057E -2	0.64239314E -3	0.12185234E -2	592
0.55293418E -3	-.44998255E -3	-.26954520E -3	0.38020979E -3	596
-.49064479E -3	-.10225473E -2	0.29908271E -3	-.55429557E -5	600
-.26056422E -5	0.24328225E -5	-.22278704E -5	0.11696672E -5	604
0.90762009E -6	0.10881101E -5	-.11508925E -5	0.11735912E -5	608
-.11591783E -5	0.35813469E -6	-.38207023E -7	0.48326146E -7	612
-.32156517E -7	0.21221186E -7	0.13102678E -6	0.25952529E -6	616
-.28509088E -6	-.67185657E -7	0.17283062E -6	-.18254255E -6	620
-.14914571E -6	-.18075244E -6	-.18396886E -6	-.13105206E -6	624
0.19649562E -7	-.48614379E -7	0.52197702E -7	0.11072716E -6	628
0.11459334E -6	-.87358993E -7	0.27579362E -3	-.27551664E -3	632
-.26545663E -3	0.10785696E -4	0.23459203E -3	-.87371959E -3	636
-.65274812E -3	0.10652986E -2	-.11017185E -2	0.12096780E -2	640
-.12179553E -2	0.12223932E -2	-.11918615E -2	-.18533187E -1	644
0.29907370E -3	0.11692399E -5	-.84160566E -6	0.58954064E -6	648
0.14990054E -5	-.16872783E -5	-.11952041E -5	0.58839979E -6	652
-.55827097E -6	0.49277262E -6	-.49319776E -6	0.33084561E -6	656
0.20119722E -6	-.21265389E -6	0.21223729E -6	-.31409979E -6	660
-.37299902E -6	0.20180553E -6	-.15332190E -6	-.10871248E -6	664
0.14020036E -6	0.24372486E -6	0.17189836E -6	0.21391349E -6	668
0.20168580E -6	0.46280799E -7	0.12654197E -6	0.24057464E -9	672
-.12382639E -6	-.14311754E -6	0.11401882E -6	0.52101943E -3	676
-.52029881E -3	-.51025777E -3	0.12487790E -3	0.47811201E -3	680
-.69661305E -3	-.35659372E -3	0.11318138E -2	-.11435830E -2	684
0.10896034E -2	-.10730869E -2	0.10530063E -2	-.13251683E -2	688
-.86396569E -2	0.39095380E -2	0.29992141E -3	0.43572306E -5	692
-.37383397E -5	0.13872875E -4	-.22175277E -5	-.13252234E -5	696
0.35138591E -6	-.29375414E -6	0.20106174E -6	-.45117508E -6	700
0.36136106E -6	0.23973273E -6	-.24775886E -6	0.24401702E -6	704
-.31336744E -6	-.34148331E -6	0.14181392E -6	-.18986351E -6	708
-.70121830E -7	0.10164330E -6	0.22326500E -6	0.13641486E -6	712
0.18085022E -6	0.18142140E -6	0.55222451E -7	0.12293902E -6	716
0.14538434E -7	-.10265342E -6	-.12223339E -6	0.98152675E -7	720
-.45744096E -3	0.45682902E -3	0.44705960E -3	-.98637398E -4	724
-.41605626E -3	0.70739766E -3	0.40194872E -3	-.10776516E -2	728
0.10936209E -2	-.10725804E -2	0.10613145E -2	-.10465614E -2	732
0.12457723E -2	0.82076430E -2	-.28708384E -2	0.14527908E -1	736
0.30036998E -3	0.32950830E -5	-.18893727E -4	0.23288175E -5	740
0.14432463E -5	-.46839083E -6	0.41105468E -6	-.31502827E -6	744
0.46531010E -6	-.35741560E -6	-.23261885E -6	0.24146977E -6	748
-.23844576E -6	0.31567940E -6	0.35096337E -6	-.15610629E -6	752
0.18246329E -6	0.78629828E -7	-.11006637E -6	-.22738426E -6	756
-.14400264E -6	-.18770696E -6	-.18514297E -6	-.53028523E -7	760
-.12320400E -6	-.11343140E -7	0.10671218E -6	0.12610206E -6	764
-.10094889E -6	0.39594926E -3	-.39543734E -3	-.38616516E -3	768
0.75963535E -4	0.35695054E -3	-.69409260E -3	-.42378798E -3	772
0.10043919E -2	-.10230183E -2	0.10272737E -2	-.10202768E -2	776
0.10099559E -2	-.11483933E -2	-.76102047E -2	0.20514453E -2	780

-.12623770E -1	0.10970081E -1	0.30258749E -3	0.32498792E -4	784
-.23647303E -5	-.15040102E -5	0.55087178E -6	-.49491827E -6	788
0.39833356E -6	-.46291671E -6	0.34425368E -6	0.22057181E -6	792
-.27979935E -6	0.22740829E -6	-.30845475E -6	-.34819833E -6	796
0.16253961E -6	-.17161181E -6	-.82832597E -7	0.11329613E -6	800
0.22395454E -6	0.14561344E -6	0.18748850E -6	0.18260956E -6	804
0.49827669E -7	0.11983760E -6	0.85892973E -8	-.10669613E -6	808
-.12537843E -6	0.10013602E -6	0.14421250E -3	-.14380418E -3	812
-.15058482E -3	0.14524006E -3	0.16956182E -3	0.77679877E -3	816
0.79756913E -3	-.54783939E -3	0.60148342E -3	-.87989379E -3	820
0.91677598E -3	-.95098401E -3	0.49181079E -3	0.57369986E -2	824
0.44304137E -2	0.48524691E -1	-.64079733E -1	0.10740296E 0	828
0.29904899E -3	0.27193629E -5	0.21973554E -5	-.15056095E -5	832
0.14678515E -5	-.13653143E -5	0.54611401E -6	-.28197406E -6	836
-.14167471E -6	0.15731516E -6	-.16133608E -6	0.30398429E -6	840
0.40247597E -6	-.27327570E -6	0.94987538E -7	0.15026577E -6	844
-.17804160E -6	-.24481441E -6	-.20122245E -6	-.23422738E -6	848
-.20334098E -6	-.28083202E -7	-.11489135E -6	0.19219071E -7	852
0.13513549E -6	0.15113620E -6	-.11835484E -6	-.11172369E -2	856
0.11153645E -2	0.11085382E -2	-.44047102E -3	-.10824828E -2	860
-.25432197E -4	-.61832189E -3	-.10216662E -2	0.96012735E -3	864
-.45684519E -3	0.37475101E -3	-.29094834E -3	0.13792643E -2	868
0.26909642E -2	-.56147726E -2	-.80462411E -2	0.84215953E -2	872
-.87916850E -2	0.90933693E -2	0.30300478E -3	-.15660943E -4	876
0.19666346E -4	-.15909050E -4	0.13348188E -4	-.77959444E -7	880
-.24895629E -6	-.25281627E -6	0.23951176E -6	-.22306637E -6	884
0.94687780E -7	-.37793342E -7	0.22276998E -6	0.22931762E -6	888
-.13452114E -6	0.12037072E -6	-.80278228E -8	0.40916950E -7	892
0.60049460E -7	0.22911984E -8	-.60998969E -7	-.40240729E -7	896
-.64488062E -7	-.37776652E -7	-.27293787E -7	0.16507558E -7	900
-.96494842E -3	0.96339009E -3	0.95478768E -3	-.34923324E -3	904
-.92439887E -3	0.24223720E -3	-.28900752E -3	-.11107126E -2	908
0.10728012E -2	-.70973954E -3	0.64749524E -3	-.58279340E -3	912
0.14103547E -2	0.35455842E -2	-.42778441E -2	-.53235438E -2	916
0.57655414E -2	-.62140757E -2	0.76178059E -2	-.51685467E -1	920
0.30439156E -3	0.32098914E -4	-.18678958E -4	0.13791904E -4	924
0.18282553E -6	-.36031131E -6	-.29815541E -6	0.29294610E -6	928
-.27953500E -6	0.22584873E -6	0.14810198E -6	0.83123842E -7	932
0.25399558E -8	-.59411606E -7	0.34712041E -7	-.11137277E -6	936
-.18496866E -8	-.43818651E -7	-.82935520E -7	-.67721015E -7	940
-.84850700E -7	-.51209588E -7	0.21175472E -7	0.37341551E -7	944
-.33347189E -7	0.75329282E -3	-.75214887E -3	-.74210637E -3	948
0.23417686E -3	0.70869095E -3	-.51433935E -3	-.75128826E -4	952
0.11501314E -2	-.11391685E -2	0.94277415E -3	-.90465701E -3	956
0.86340536E -3	-.13746284E -2	-.42945004E -2	0.27924548E -2	960
0.25451537E -2	-.30565613E -2	0.35871846E -2	-.64492666E -2	964
0.70354779E -1	0.10545271E 0	0.24927674E -3	0.52238446E -5	968
-.50538881E -5	-.45694993E -6	0.45960959E -6	0.32922447E -6	972
-.33318597E -6	0.32392333E -6	-.35514136E -6	-.34196391E -6	976
0.74198655E -7	-.26540838E -6	-.25579526E -7	0.60233194E -7	980
0.21686882E -6	0.10229722E -6	0.15379579E -6	0.17026241E -6	984
0.71076404E -7	0.12820742E -6	0.33888221E -7	-.83682353E -7	988
-.10505571E -6	0.85285862E -7	-.70475372E -3	0.70369709E -3	992
0.69367437E -3	-.21168810E -3	-.66058485E -3	0.54179496E -3	996
0.12656673E -3	-.11279348E -2	0.11211625E -2	-.95388108E -3	1000
0.92020476E -3	-.88337187E -3	0.13352956E -2	0.42628941E -2	1004
-.24833810E -2	-.20517108E -2	0.25502899E -2	-.30683380E -2	1008
0.59781486E -2	-.56903995E -1	-.63205568E -1	0.17454897E -1	1012
0.29917679E -3	0.49889378E -5	0.49045256E -6	-.46681108E -6	1016
-.32829222E -6	0.33354686E -6	-.32505293E -6	0.36838280E -6	1020
0.36503196E -6	-.96313461E -7	0.26247094E -6	0.37706368E -7	1024
-.73182055E -7	-.22841577E -6	-.11525437E -6	-.16717069E -6	1028
-.17990643E -6	-.70198200E -7	-.13214502E -6	-.30667188E -7	1032
0.91291438E -7	0.11301421E -6	-.91276808E -7	0.65624888E -3	1036
-.65527882E -3	-.64531045E -3	0.18996961E -3	0.61264756E -3	1040
-.56586737E -3	-.17481049E -3	0.11029401E -2	-.11001554E -2	1044

0.96100112E -3	-.93164337E -3	0.89911923E -3	-.12933847E -2	1048
-.42174327E -2	0.21892995E -2	0.15966376E -2	-.20821724E -2	1052
0.25873736E -2	-.55289280E -2	0.48531570E -1	0.48144509E -1	1056
-.17178076E -1	0.16675551E -1	0.29908904E -3	-.52117135E -6	1060
0.47221265E -6	0.32630516E -6	-.33278818E -6	0.32506275E -6	1064
-.37991699E -6	-.38604163E -6	0.11727266E -6	-.25876952E -6	1068
-.49212178E -7	0.85354135E -7	0.23871518E -6	0.12728764E -6	1072
0.17942479E -6	0.18853273E -6	0.69119011E -7	0.13546728E -6	1076
0.27478953E -7	-.98263372E -7	-.12024511E -6	0.96696329E -7	1080
-.96052978E -3	0.95893357E -3	0.95172945E -3	-.36975368E -3	1084
-.92543776E -3	0.42656465E -4	-.42240810E -3	-.81352334E -3	1088
0.76996722E -3	-.41071356E -3	0.35111529E -3	-.29019984E -3	1092
0.10108214E -2	0.10967355E -2	-.16304339E -2	-.16235846E -2	1096
0.16691543E -2	-.17095107E -2	0.17945316E -2	0.59029505E -4	1100
0.94914497E -3	-.15853952E -2	0.16683979E -2	-.17425291E -2	1104
0.29919561E -3	-.55429981E -5	-.25711594E -5	0.24028474E -5	1108
-.22008173E -5	0.11951639E -5	0.15017570E -6	0.98726930E -6	1112
0.12335795E -5	-.29784535E -6	0.24397936E -6	-.94625190E -7	1116
0.15435217E -6	0.68223172E -7	-.48234460E -7	-.12985604E -6	1120
-.11006938E -6	-.12277157E -6	-.38990344E -7	-.15545217E -7	1124
0.31841785E -8	0.54697192E -3	-.54599945E -3	-.54477948E -3	1128
0.24405581E -3	0.53818207E -3	0.23292336E -3	0.47619052E -3	1132
0.24496763E -3	-.20656387E -3	-.61580291E -4	0.10270136E -3	1136
-.14361880E -3	-.37426048E -3	-.26370113E -4	0.10484488E -2	1140
0.12296332E -2	-.12061914E -2	0.11783386E -2	-.86496978E -3	1144
-.11551699E -2	-.15027062E -2	0.15607194E -2	-.15557372E -2	1148
0.15465537E -2	-.18526335E -1	0.29898256E -3	0.11745097E -5	1152
-.87502571E -6	0.66515785E -6	0.14177541E -5	0.29174374E -5	1156
-.17049137E -5	-.49964139E -6	0.39022771E -6	-.38919520E -6	1160
-.27119197E -6	-.35191149E -6	-.33060381E -6	-.19632589E -6	1164
0.41363775E -7	-.62605903E -7	0.87696971E -7	0.15401747E -6	1168
0.15539882E -6	-.10909113E -6	0.30880383E -3	-.30822788E -3	1172
-.30875832E -3	0.15199348E -3	0.30857451E -3	0.23936702E -3	1176
0.36737037E -3	0.46568972E -4	-.19256667E -4	-.15834177E -3	1180
0.18448759E -3	-.21011771E -3	-.12672333E -3	0.22482652E -3	1184
0.63279961E -3	0.80525640E -3	-.77260665E -3	0.73695669E -3	1188
-.42000021E -3	-.10727299E -2	-.12033402E -2	0.11182066E -2	1192
-.10950625E -2	0.10699468E -2	-.85207958E -2	0.39283555E -2	1196
0.30039640E -3	0.37673316E -5	-.29767880E -5	0.13259176E -4	1200
0.13452530E -4	-.21658091E -5	-.16318495E -6	0.34769861E -6	1204
-.36063212E -6	-.32686653E -6	-.34243074E -6	-.34149875E -6	1208
-.22416839E -6	0.80679097E -8	-.97661694E -7	0.59539994E -7	1212
0.15240029E -6	0.16009178E -6	-.11364238E -6	-.34217453E -3	1216
0.34154464E -3	0.34175054E -3	-.16401032E -3	-.34043751E -3	1220
-.23153738E -3	-.37599813E -3	-.79690560E -4	0.51227866E -4	1224
0.13702795E -3	-.16502613E -3	0.19257631E -3	0.16588126E -3	1228
-.17469809E -3	-.68118197E -3	-.84795197E -3	0.81793398E -3	1232
-.78478858E -3	0.47966692E -3	0.10455630E -2	0.12069064E -2	1236
-.11486012E -2	0.11292179E -2	-.11075705E -2	0.80836656E -2	1240
-.29759444E -2	0.12541201E -1	0.30111214E -3	0.23488380E -5	1244
-.18110198E -4	-.16159180E -4	0.22696794E -5	0.27051311E -6	1248
-.35751168E -6	0.36699937E -6	0.31573926E -6	0.34398340E -6	1252
0.33836157E -6	0.21717516E -6	-.14331898E -7	0.90195957E -7	1256
-.64353998E -7	-.15105667E -6	-.15743405E -6	0.11132583E -6	1260
0.35055105E -3	-.34991039E -3	-.34990708E -3	0.16555503E -3	1264
0.34793942E -3	0.21830702E -3	0.36775501E -3	0.97353134E -4	1268
-.69206826E -4	-.11884594E -3	0.14697963E -3	-.17472345E -3	1272
-.18415272E -3	0.13728281E -3	0.68613423E -3	0.84318672E -3	1276
-.81589400E -3	0.78551031E -3	-.49891792E -3	-.99014851E -3	1280
-.11644893E -2	0.11248949E -2	-.11085414E -2	0.10898267E -2	1284
-.74620590E -2	0.22868801E -2	-.10007361E -1	0.78005425E -2	1288
0.30380131E -3	0.31444137E -4	0.20006984E -4	-.22818746E -5	1292
-.33405222E -6	0.35167689E -6	-.35861702E -6	-.29801882E -6	1296
-.33326742E -6	-.32483363E -6	-.20529580E -6	0.17760220E -7	1300
-.82400610E -7	0.65140341E -7	0.14496140E -6	0.15029145E -6	1304
-.10598656E -6	-.67556994E -3	0.67440250E -3	0.67132779E -3	1308

-.28359351E -3	-.65862308E -3	-.15025242E -3	-.45962632E -3	1312
-.41350606E -3	0.37366638E -3	-.79375720E -4	0.32900796E -4	1316
0.13814962E -4	0.55994660E -3	0.32931928E -3	-.11698469E -2	1320
-.12814150E -2	0.12777620E -2	-.12696763E -2	0.10817838E -2	1324
0.78400501E -3	0.12556509E -2	-.14659137E -2	0.14842037E -2	1328
-.14972600E -2	0.57170663E -2	0.41528526E -2	0.45928996E -1	1332
-.60951740E -1	0.10350231E 0	0.30021440E -3	-.19315874E -4	1336
0.27507522E -5	0.13867316E -5	-.43270783E -6	0.40714298E -6	1340
0.18062091E -6	0.33796233E -6	0.28585205E -6	0.13446254E -6	1344
-.77846056E -7	0.92246048E -8	-.11059931E -6	-.13067673E -6	1348
-.12342506E -6	0.83246856E -7	-.87938586E -3	0.87790669E -3	1352
0.87205045E -3	-.34775270E -3	-.85012740E -3	-.32443497E -4	1356
-.44786728E -3	-.67409388E -3	0.63094041E -3	-.28913357E -3	1360
0.23333666E -3	-.17662904E -3	0.85182283E -3	0.78418769E -3	1364
-.14273099E -2	-.14570512E -2	0.14816244E -2	-.15013556E -2	1368
0.14759322E -2	0.34392150E -3	0.10366255E -2	-.14806508E -2	1372
0.15341248E -2	-.15803991E -2	0.23573575E -2	0.95388515E -2	1376
0.49069736E -1	-.58100346E -1	0.72514858E -1	-.64340266E -1	1380
0.29902257E -3	0.26653926E -5	0.23876014E -5	-.41808972E -6	1384
0.36027320E -6	-.51031697E -8	0.25669696E -6	0.16621170E -6	1388
0.16278072E -7	-.12871026E -6	-.79472522E -7	-.13575563E -6	1392
-.80240622E -7	-.60047763E -7	0.35096549E -7	0.54432704E -3	1396
-.54346650E -3	-.53729388E -3	0.18608548E -3	0.51632464E -3	1400
-.20046310E -3	0.72316188E -4	0.59673823E -3	-.58192532E -3	1404
0.42766799E -3	-.39971413E -3	0.37040120E -3	-.68601642E -3	1408
-.93169447E -3	0.72697117E -3	0.59259124E -3	-.64469809E -3	1412
0.69508635E -3	-.96125770E -3	0.65444277E -3	0.14874650E -3	1416
0.32621122E -3	-.39485043E -3	0.45872275E -3	0.29422915E -2	1420
-.57575318E -2	-.82377566E -2	0.86726113E -2	-.91058030E -2	1424
0.97361144E -2	0.89136840E -2	0.30007154E -3	-.17364516E -4	1428
0.60494259E -7	0.23895508E -7	0.40415660E -6	0.11928679E -6	1432
0.21794213E -6	0.23853440E -6	0.13150088E -6	0.19785742E -6	1436
0.80346718E -7	-.83351976E -7	-.11323963E -6	0.86973399E -7	1440
-.19578496E -3	0.19539626E -3	0.19679135E -3	-.10874949E -3	1444
-.19974804E -3	-.24404563E -3	-.31683899E -3	0.49119234E -4	1448
-.71155552E -4	0.20560437E -3	-.22458063E -3	0.24288634E -3	1452
0.79830426E -5	-.33275981E -3	-.42525973E -3	-.58575142E -3	1456
0.55011427E -3	-.51224952E -3	0.20601917E -3	0.98096380E -3	1460
0.10101487E -2	-.86751652E -3	0.83747689E -3	-.80656853E -3	1464
0.41969651E -2	-.20923405E -2	-.16461633E -2	0.21542807E -2	1468
-.26836083E -2	0.56962843E -2	0.85005025E -2	-.57867923E -1	1472
0.29897087E -3	-.41479335E -6	0.43443461E -6	0.44116882E -6	1476
0.41462622E -6	0.41976811E -6	0.27836928E -6	0.11345150E -7	1480
0.13481577E -6	-.51246275E -7	-.17191087E -6	-.18401899E -6	1484
0.12953173E -6	-.34446115E -3	0.34392427E -3	0.33958793E -3	1488
-.11351899E -3	-.32507312E -3	0.15389281E -3	-.16289339E -4	1492
-.38820615E -3	0.38066238E -3	-.29432522E -3	0.27809073E -3	1496
-.26093111E -3	0.43348350E -3	0.56408318E -3	-.37565739E -3	1500
-.28026686E -3	0.31176946E -3	-.34242610E -3	0.51447832E -3	1504
-.42483911E -3	-.15857023E -3	-.10352543E -3	0.14188612E -3	1508
-.17768769E -3	-.92644263E -3	0.13120385E -2	0.13181249E -2	1512
-.13641802E -2	0.14062712E -2	-.15253489E -2	-.13881912E -2	1516
0.12131328E -3	-.13874039E -2	0.24906411E -3	-.76828517E -6	1520
-.17608772E -5	-.63769567E -6	-.69699962E -6	-.43449330E -6	1524
-.19671519E -6	-.32066334E -6	-.10267887E -6	0.13456543E -6	1528
0.17083909E -6	-.11675421E -6	0.42153916E -3	-.42086895E -3	1532
-.41615830E -3	0.14588332E -3	0.40012542E -3	-.13672925E -3	1536
0.66935638E -4	0.43139684E -3	-.41952383E -3	0.30157776E -3	1540
-.28035931E -3	0.25819981E -3	-.49062591E -3	-.58748397E -3	1544
0.47719405E -3	0.38967067E -3	-.42014753E -3	0.44939436E -3	1548
-.60193859E -3	0.35189256E -3	0.51754280E -4	0.22256978E -3	1552
-.26143763E -3	0.29734906E -3	0.74313805E -3	-.13009448E -2	1556
-.13425936E -2	0.13733580E -2	-.13999311E -2	0.14207224E -2	1560
0.11882081E -2	0.16403224E -3	0.14495360E -2	-.25689648E -2	1564
0.29908124E -3	0.26104631E -5	0.59828717E -6	0.70923744E -6	1568
0.44075526E -6	0.24347709E -6	0.35263299E -6	0.14660482E -6	1572

-.11096072E -6	-.15197951E -6	0.10479545E -6	0.62922428E -3	1576
-.62816398E -3	-.62384709E -3	0.24931094E -3	0.60779125E -3	1580
0.27572167E -4	0.31183554E -3	0.45058616E -3	-.42095977E -3	1584
0.18895599E -3	-.15110814E -3	0.11270602E -3	-.55829449E -3	1588
-.43860335E -3	0.80103889E -3	0.79658641E -3	-.80768417E -3	1592
0.81608543E -3	-.79490967E -3	-.19119896E -3	-.53757846E -3	1596
0.75526414E -3	-.77959029E -3	0.80019182E -3	-.38363409E -3	1600
-.86982456E -3	-.11048761E -2	0.10570295E -2	-.10052765E -2	1604
0.55764124E -3	-.60159885E -4	0.14315414E -2	0.14548039E -2	1608
-.58655366E -2	0.87282745E -2	0.29998583E -3	-.17372908E -4	1612
-.13433446E -5	-.60074300E -7	0.31200196E -6	0.18451286E -6	1616
0.30709953E -6	0.15250280E -6	0.11349964E -6	-.57590972E -7	1620
0.53468109E -3	-.53381212E -3	-.52868037E -3	0.19495362E -3	1624
0.51079115E -3	-.10008603E -3	0.15130664E -3	0.48428637E -3	1628
-.46552366E -3	0.29889295E -3	-.27013102E -3	0.24044942E -3	1632
-.56432531E -3	-.59742279E -3	0.62422925E -3	0.55368989E -3	1636
-.58087026E -3	0.60623886E -3	-.71833522E -3	0.21273836E -3	1640
-.12858460E -3	0.40939339E -3	-.44709602E -3	0.48132992E -3	1644
0.42582214E -3	-.12192905E -2	-.13119943E -2	0.13185493E -2	1648
-.13208358E -2	0.11953993E -2	0.83430027E -3	0.56061631E -3	1652
0.14652381E -2	-.24668527E -2	0.25058925E -2	-.57912431E -1	1656
0.29905680E -3	0.73929975E -6	0.43407886E -6	0.34401888E -6	1660
0.41174113E -6	0.24195554E -6	-.54130970E -7	-.10370961E -6	1664
0.74971574E -7	0.60541523E -3	-.60441790E -3	-.59919832E -3	1668
0.22774719E -3	0.58065735E -3	-.61758582E -4	0.21768652E -3	1672
0.50350623E -3	-.47970303E -3	0.27942510E -3	-.24562002E -3	1676
0.21096793E -3	-.59722134E -3	-.57227623E -3	0.71542445E -3	1680
0.66288809E -3	-.68568670E -3	0.70627971E -3	-.77746053E -3	1684
0.82933440E -4	-.27407269E -3	0.54460269E -3	-.57916248E -3	1688
0.61001532E -3	0.15931161E -3	-.10923827E -2	-.12197811E -2	1692
0.12080917E -2	-.11923464E -2	0.96547202E -3	0.52905615E -3	1696
0.81246798E -3	0.14002651E -2	-.23446784E -2	0.24042779E -2	1700
-.43348616E -2	0.24721048E -2	0.29903927E -3	0.38915269E -6	1704
0.46441650E -6	0.46639760E -6	0.35574760E -6	0.18616728E -7	1708
-.38882620E -7	0.36905507E -7	0.55966347E -3	-.56870698E -3	1712
-.56457062E -3	0.22372195E -3	0.54937259E -3	0.10823530E -4	1716
0.26536628E -3	0.41039116E -3	-.38525532E -3	0.18286169E -3	1720
-.14965473E -3	0.11591862E -3	-.50140372E -3	-.39367604E -3	1724
0.66576841E -3	0.64995148E -3	-.66072792E -3	0.66930136E -3	1728
-.66488366E -3	-.11802135E -3	-.40575050E -3	0.59308522E -3	1732
-.61435923E -3	0.63253927E -3	-.20288939E -3	-.63903729E -3	1736
-.77004690E -3	0.74079708E -3	-.70895104E -3	0.43115806E -3	1740
0.33757762E -4	0.84715560E -3	0.92328216E -3	-.14436003E -2	1744
0.14686793E -2	-.11055650E -3	0.14482760E -2	0.13013431E -2	1748
0.29901489E -3	0.24619362E -5	0.15698698E -5	0.12888289E -5	1752
0.26774021E -6	0.17306671E -6	-.64035231E -7	0.89554371E -4	1756
-.89381543E -4	-.89754860E -4	0.47013818E -4	0.90333702E -4	1760
0.86903223E -4	0.11993120E -3	-.53047923E -5	0.13696755E -4	1764
-.66228968E -4	0.73730440E -4	-.81005658E -4	-.17057865E -4	1768
0.85148336E -4	0.13882303E -3	0.17853370E -3	-.16879479E -3	1772
0.15842231E -3	-.74672091E -4	-.25350841E -3	-.26770793E -3	1776
0.23614270E -3	-.22862874E -3	0.22084386E -3	-.42923802E -3	1780
0.15694022E -3	0.77595771E -4	-.10535535E -3	0.13292212E -3	1784
-.30064313E -3	-.43854252E -3	0.43000342E -3	0.22879617E -4	1788
-.63121260E -3	0.78629673E -3	0.11143174E -2	0.11345827E -2	1792
0.15423139E -2	0.82334906E -2	0.30022738E -3	-.19337487E -4	1796
0.57418012E -6	0.40448182E -6	0.38790335E -6	-.17024912E -6	1800
0.33854100E -3	-.33795590E -3	-.33625175E -3	0.14172288E -3	1804
0.32940621E -3	0.69536403E -4	0.21536439E -3	0.19183445E -3	1808
-.17340023E -3	0.38059008E -4	-.16680427E -4	-.47818444E -5	1812
-.25127606E -3	-.12324186E -3	0.41759724E -3	0.43920426E -3	1816
-.43706492E -3	0.43341531E -3	-.36929835E -3	-.24330930E -3	1820
-.38941326E -3	0.45431009E -3	-.45875210E -3	0.46153436E -3	1824
-.41141103E -3	-.17913866E -3	-.29735068E -3	0.26329304E -3	1828
-.22818698E -3	-.23794742E -4	-.30873461E -3	0.70572787E -3	1832
0.43926642E -3	-.10916099E -2	0.12094200E -2	0.77391957E -3	1836

0.14384177E -2	0.16521608E -2	0.57804541E -2	-.64409472E -1	1840
0.29896193E -3	0.17401184E -5	0.45962343E -6	0.38935478E -6	1844
-.16069767E -6	-.43760537E -4	0.43710773E -4	0.42309320E -4	1848
-.46747491E -5	-.37999608E -4	0.88975831E -4	0.62218589E -4	1852
-.10482607E -3	0.10746827E -3	-.11390846E -3	0.11378549E -3	1856
-.11329683E -3	0.10291610E -3	0.18295921E -3	-.13369079E -4	1860
0.32029577E -4	-.19303837E -4	0.64415467E -5	0.79962650E -4	1864
-.23631660E -3	-.18300796E -3	0.10578706E -3	-.92926650E -4	1868
0.80518051E -4	-.39351390E -3	0.30621812E -3	0.25380799E -3	1872
-.27543024E -3	0.29626265E -3	-.40429350E -3	-.45473926E -3	1876
0.24633358E -3	-.18351647E -3	-.31750176E -3	0.46148753E -3	1880
0.10695659E -2	0.78933665E -3	0.11714599E -2	0.42608590E -2	1884
0.22873818E -2	0.58205352E -2	0.29900729E -3	0.44431702E -6	1888
0.44462300E -6	-.18151038E -6	-.39480175E -3	0.39415286E -3	1892
0.39060820E -3	-.14756255E -3	-.37810968E -3	0.45633516E -4	1896
-.13301923E -3	-.32358368E -3	0.30876677E -3	-.18358293E -3	1900
0.16233739E -3	-.14054515E -3	0.37814231E -3	0.34722910E -3	1904
-.41517374E -3	-.37740109E -3	0.39111882E -3	-.40356671E -3	1908
0.45013215E -3	-.63149088E -4	0.14022542E -3	-.29663914E -3	1912
0.31661225E -3	-.33445711E -3	-.99551269E -4	0.51152420E -3	1916
0.54829565E -3	-.54362604E -3	0.53715766E -3	-.44482098E -3	1920
-.25805410E -3	-.31450772E -3	-.57566835E -3	0.45597628E -3	1924
-.38005795E -3	0.49397267E -3	-.18736926E -3	0.54731230E -4	1928
0.86909673E -3	0.14428756E -2	0.15287487E -2	0.14859739E -2	1932
0.29904409E -3	0.72704609E -6	-.99512466E -7	-.44715839E -3	1936
0.44641749E -3	0.44266989E -3	-.17026134E -3	-.42928845E -3	1940
0.29309446E -4	-.17096981E -3	-.34769635E -3	0.32979973E -3	1944
-.18282857E -3	0.15820857E -3	-.13305183E -3	0.41137147E -3	1948
0.35420888E -3	-.47634215E -3	-.44410965E -3	0.45667670E -3	1952
-.46774277E -3	0.49937365E -3	-.12782629E -4	0.20826566E -3	1956
-.36906932E -3	0.38882470E -3	-.40623982E -3	-.20095676E -4	1960
0.51219549E -3	0.56644875E -3	-.55604090E -3	0.54377103E -3	1964
-.41308828E -3	-.18818718E -3	-.41478762E -3	-.61329980E -3	1968
0.57343545E -3	-.51347862E -3	0.35252802E -3	-.35136324E -3	1972
-.13679777E -3	0.55217473E -3	0.13690067E -2	0.12899839E -2	1976
0.14833345E -2	0.24312338E -2	0.29902802E -3	-.35666411E -7	1980
0.36834044E -3	-.36772818E -3	-.36470270E -3	0.14121707E -3	1984
0.35385846E -3	-.16895225E -4	0.14601341E -3	0.27723630E -3	1988
-.26228808E -3	0.14097791E -3	-.12074739E -3	0.10010973E -3	1992
-.32836990E -3	-.26938364E -3	0.37907999E -3	0.35550414E -3	1996
-.36444997E -3	0.37218892E -3	-.39067945E -3	-.75710459E -5	2000
-.17787513E -3	0.29882371E -3	-.31335996E -3	0.32607930E -3	2004
-.11070426E -4	-.35967172E -3	-.40045456E -3	0.39124605E -3	2008
-.38072543E -3	0.27761439E -3	0.10906839E -3	0.31573790E -3	2012
0.43201908E -3	-.39094322E -3	0.35264486E -3	-.17504583E -3	2016
0.25248996E -3	0.12616795E -3	-.20343636E -3	-.59965664E -3	2020
-.53332301E -3	-.60637197E -3	-.33247856E -3	-.11927448E -3	2024
0.29902534E -3				
-.28359351E -3				

0.30448909E -3	0.66489464E -6	-.27804511E -6	-.30376638E -6	4
-.12549744E -8	-.23876581E -7	-.27747999E -7	0.25227392E -7	8
-.76110890E -8	0.66940255E -8	-.48404383E -8	0.31393235E -7	12
0.17152680E -7	-.24868627E -7	-.21017878E -7	0.21081051E -7	16
-.20536822E -7	0.18904172E -7	-.20050680E -9	0.86044006E -8	20
-.16711695E -7	0.17718038E -7	-.18385637E -7	0.17883292E -8	24
0.12383451E -7	0.13114829E -7	-.12556708E -7	0.11795190E -7	28
-.58449145E -8	0.52362327E -9	-.11635246E -7	-.13678385E -7	32
0.11082396E -7	-.10015803E -7	0.26399002E -8	-.76460286E -8	36
-.45801579E -8	0.18437253E -8	0.90240872E -8	0.70452300E -8	40
0.94799998E -8	0.51028894E -9	0.30098769E -8	-.21607663E -8	44
0.21836403E -2	0.29836037E -3	-.38199552E -5	0.21955221E -6	48
-.82752157E -7	-.13480925E -7	0.11145807E -6	-.10332628E -6	52
0.73821741E -7	-.74278460E -7	0.70718628E -7	-.11028269E -6	56
-.61465798E -7	0.62341408E -7	0.48724450E -7	-.49872953E -7	60
0.49326024E -7	-.53460154E -7	0.14821512E -7	-.11290579E -7	64
0.37001578E -7	-.34753365E -7	0.41979741E -7	0.40997126E -8	68
-.37852657E -7	-.37496149E -7	0.35262580E -7	-.34290253E -7	72
0.20742491E -7	0.53437705E -8	0.25488717E -7	0.38189543E -7	76
-.27030105E -7	0.23751002E -7	-.10934233E -7	0.17119588E -7	80
0.85165383E -8	-.77721294E -8	-.24313968E -7	-.20265185E -7	84
-.24866984E -7	-.11925693E -7	-.62493666E -8	0.47736148E -8	88
-.88519547E -3	-.12803159E -1	0.29915693E -3	-.18620884E -4	92
0.94001170E -7	-.28467140E -6	-.49693098E -6	0.45436417E -6	96
-.19344299E -6	0.18247578E -6	-.15441258E -6	0.53901640E -6	100
0.29404374E -6	-.39107022E -6	-.32519081E -6	0.32747440E -6	104
-.31998412E -6	0.30509467E -6	-.21904748E -7	0.12169908E -6	108
-.25610439E -6	0.27220767E -6	-.28341364E -6	0.16812758E -7	112
0.20304182E -6	0.21162128E -6	-.20308789E -6	0.19106693E -6	116
-.99584660E -7	-.49684180E -9	-.17885091E -6	-.21946992E -6	120
0.17260127E -6	-.15509624E -6	0.46984238E -7	-.11706671E -6	124
-.67848328E -7	0.32980879E -7	0.14348396E -6	0.11369576E -6	128
0.14984391E -6	0.78737796E -7	0.45416019E -7	-.32972369E -7	132
-.10543319E -2	-.61064171E -4	-.62244536E -1	0.29833867E -3	136
0.16238319E -6	0.32022296E -7	-.21052214E -6	0.19525172E -6	140
-.14189875E -6	0.14298140E -6	-.13646687E -6	0.20652166E -6	144
0.11439200E -6	-.11429045E -6	-.88935768E -7	0.91128032E -7	148
-.90197673E -7	0.98513370E -7	-.28435026E -7	0.19752421E -7	152
-.67324486E -7	0.72383622E -7	-.76509541E -7	-.82641749E -8	156
0.69804021E -7	0.68949116E -7	-.66708872E -7	0.63097871E -7	160
-.38458754E -7	-.10363383E -7	-.48167623E -7	-.70138408E -7	164
0.49301378E -7	-.43256164E -7	0.20354027E -7	-.31080511E -7	168
-.15286551E -7	0.14470535E -7	0.44537879E -7	0.37227911E -7	172
0.45492412E -7	0.21689454E -7	0.11290359E -7	-.86535665E -8	176
-.26639177E -5	-.27330015E -3	0.34809943E -3	0.54429147E -3	180
0.30029491E -3	-.19339655E -4	-.18230964E -5	0.16217157E -5	184
-.51567602E -6	0.45216243E -6	-.33488457E -6	0.11901399E -5	188
0.38341597E -6	-.53611128E -6	-.44195514E -6	0.44307476E -6	192
-.43136427E -6	0.39904227E -6	-.83429210E -8	0.17176470E -6	196
-.33774887E -6	0.35792406E -6	-.37129834E -6	0.30644199E -7	200
0.22901171E -6	0.24060658E -6	-.23039966E -6	0.21642730E -6	204
-.10821450E -6	0.77696843E -8	-.20974599E -6	-.24825186E -6	208
0.19464661E -6	-.17533171E -6	0.47232905E -7	-.13302447E -6	212
-.78944708E -7	0.32530914E -7	0.15545664E -6	0.12167469E -6	216
0.16280698E -6	0.86617242E -7	0.50832503E -7	-.36237779E -7	220
-.79217292E -4	-.74961155E -4	-.92247947E -3	0.14238896E -3	224
-.64402210E -1	0.29891499E -3	-.25802153E -5	0.23264384E -5	228
-.14713691E -5	0.14335592E -5	-.13141677E -5	0.14720777E -5	232
0.47413161E -6	-.50691303E -6	-.39481092E -6	0.40160146E -6	236
-.34525661E -6	0.41207600E -6	-.87697381E -7	0.10504875E -6	240
-.29181963E -6	0.31216668E -6	-.32788353E -6	-.14342442E -7	244
0.25011214E -6	0.24992651E -6	-.24112032E -6	0.22762188E -6	248
-.13240048E -6	-.25835861E -7	-.18455493E -6	-.25332156E -6	252

0.17983708E -6	-.15873623E -6	0.65117056E -7	-.11558923E -6	256
-.60297903E -7	0.45462256E -7	0.15440840E -6	0.12685662E -6	260
0.15855125E -6	0.77573567E -7	0.41872587E -7	-.31176676E -7	264
-.94859795E -4	0.34995192E -3	-.17108769E -2	-.70111474E -3	268
-.66269356E -2	-.86319364E -2	0.30154031E -3	0.18218431E -5	272
-.18572941E -4	0.13841774E -4	-.11429693E -4	-.43119164E -6	276
-.70947311E -7	0.38332953E -6	0.35392179E -6	-.34409955E -6	280
0.32706013E -6	-.21828394E -6	-.13666432E -6	-.22117258E -6	284
0.28271630E -6	-.29427312E -6	0.29791345E -6	-.94464685E -7	288
-.93289809E -7	-.11970351E -6	0.11154288E -6	-.10284493E -6	292
0.19677130E -7	-.61742246E -7	0.16073957E -6	0.13066505E -6	296
-.13280320E -6	0.12498689E -6	0.39677917E -8	0.10280134E -6	300
0.74847270E -7	0.37699589E -8	-.87018324E -7	-.58072993E -7	304
-.96326820E -7	-.62380246E -7	-.42672991E -7	0.28111996E -7	308
0.86846285E -4	-.32688312E -3	0.15751887E -2	0.65465176E -3	312
0.59416873E -2	0.78350956E -2	0.60417897E -2	0.30447813E -3	316
0.32028365E -4	-.18601409E -4	0.13785187E -4	0.47599033E -6	320
0.79412518E -7	-.37416097E -6	-.34329080E -6	0.33421337E -6	324
-.31800405E -6	0.21608422E -6	0.12627135E -6	0.21066821E -6	328
-.27331177E -6	0.28469234E -6	-.28850992E -6	0.88353700E -7	332
0.93726924E -7	0.11850476E -6	-.11062618E -6	0.10212846E -6	336
-.21712234E -7	0.57338363E -7	-.15540307E -6	-.12884100E -6	340
0.12897006E -6	-.12111917E -6	-.21566258E -8	-.99250301E -7	344
-.71682722E -7	-.24491708E -8	0.85335366E -7	0.57519030E -7	348
0.94162989E -7	0.60369158E -7	0.41029475E -7	-.27109881E -7	352
-.41983117E -4	0.29421848E -3	-.95125016E -3	-.58604186E -3	356
-.30674319E -2	-.62782196E -2	-.62229100E -1	0.10519102E 0	360
0.29932388E -3	0.51799335E -5	-.51785614E -5	-.17152322E -5	364
-.32859347E -6	0.58668264E -6	0.49718643E -6	-.49326168E -6	368
0.47633566E -6	-.40133395E -6	-.57197922E -7	-.22950912E -6	372
0.38060425E -6	-.40074781E -6	0.41218893E -6	-.65356399E -7	376
-.20463221E -6	-.22465139E -6	0.21367581E -6	-.19980498E -6	380
0.85207930E -7	-.34024210E -7	0.22139626E -6	0.23436624E -6	384
-.19622273E -6	0.17911403E -6	-.30803918E -7	0.13942026E -6	388
0.89053171E -7	-.20603577E -7	-.14691080E -6	-.11034462E -6	392
-.15616622E -6	-.88051579E -7	-.54459211E -7	0.37669747E -7	396
0.34356427E -4	-.28151786E -3	0.83515431E -3	0.56016895E -3	400
0.25357482E -2	0.57805702E -2	0.47482200E -1	-.62913101E -1	404
0.17305113E -1	0.29921918E -3	0.52854162E -5	0.19087476E -5	408
0.35987067E -6	-.60783180E -6	-.51104646E -6	0.50794675E -6	412
-.49121060E -6	0.42154014E -6	0.45919885E -7	0.22805413E -6	416
-.38943941E -6	0.41050186E -6	-.42285397E -6	0.60680003E -7	420
0.21696382E -6	0.23557251E -6	-.22465607E -6	0.21026590E -6	424
-.92890050E -7	0.29913445E -7	-.22673650E -6	-.24527574E -6	428
0.20218373E -6	-.18405062E -6	0.35038115E -7	-.14252385E -6	432
-.89811095E -7	0.23579735E -7	0.15298966E -6	0.11597859E -6	436
0.16211630E -6	0.90327940E -7	0.55322942E -7	-.38450804E -7	440
-.26565682E -4	0.26788069E -3	-.71567422E -3	-.53243806E -3	444
-.20163865E -2	-.52891067E -2	-.49093246E -1	0.48205993E -1	448
-.17606541E -1	0.17664029E -1	0.29910906E -3	-.21093861E -5	452
-.39093159E -6	0.62756415E -6	0.52358622E -6	-.52134783E -6	456
0.50486471E -6	-.44093083E -6	-.34145557E -7	-.22577827E -6	460
0.39721299E -6	-.41915011E -6	0.43233843E -6	-.55673754E -7	464
-.22890219E -6	-.24660633E -6	0.23518001E -6	-.22830463E -6	468
0.10046769E -6	-.25592587E -7	0.23146199E -6	0.25565660E -6	472
-.20762707E -6	0.18850318E -6	-.39264818E -7	0.14523398E -6	476
0.90290480E -7	-.26553182E -7	-.15871559E -6	-.12116902E -6	480
-.16767985E -6	-.92362663E -7	-.56025393E -7	0.39124288E -7	484
0.10571545E -3	-.33868373E -3	0.18273017E -2	0.67666832E -3	488
0.42136636E -2	0.47899949E -2	-.23816438E -2	0.29073693E -2	492
-.59649119E -2	0.65036657E -2	-.70522308E -2	0.29915213E -3	496
-.38304464E -7	-.44653577E -6	-.42833859E -6	0.41072634E -6	500
-.38597310E -6	0.21251313E -6	0.23157684E -6	0.30238868E -6	504
-.34050021E -6	0.35177587E -6	-.35249243E -6	0.13719669E -6	508
0.65899633E -7	0.10297478E -6	-.33799283E -7	0.35094669E -7	512
0.68701161E -8	0.93384769E -7	-.17693351E -6	-.11706529E -6	516

0.13742467E -6	-.13186183E -6	-.21790352E -7	-.11202273E -6	520
-.87400061E -7	-.16405913E -7	0.80168110E -7	0.47605972E -7	524
0.91740096E -7	0.65419772E -7	0.47477911E -7	-.30287849E -7	528
0.58032335E -4	-.19180044E -3	0.10042298E -2	0.38024729E -3	532
0.13673295E -2	0.15619571E -2	-.41498610E -3	0.51533476E -3	536
-.11423738E -2	0.12266186E -2	-.13078898E -2	-.12804343E -3	540
0.29931762E -3	-.56778666E -5	-.26870395E -5	0.25231006E -5	544
-.23222414E -5	0.13266724E -5	0.76936761E -6	0.10374757E -5	548
-.11977064E -5	0.12282103E -5	-.12228604E -5	0.23915380E -6	552
0.14181437E -6	0.20139641E -6	-.18472745E -6	0.16837354E -6	556
-.29740450E -8	0.15395095E -6	-.30840631E -6	-.21710309E -6	560
0.22057877E -6	-.20883682E -6	-.25762192E -7	-.17337449E -6	564
-.13107220E -6	-.18945835E -7	0.12444913E -6	0.76894873E -7	568
0.14005690E -6	0.94919401E -7	0.67472953E -7	-.42434804E -7	572
-.82135945E -4	0.18487034E -3	-.13010786E -2	-.36855393E -3	576
-.18311511E -2	-.16033167E -2	0.15697809E -2	-.16498640E -2	580
0.20015452E -2	-.20354770E -2	0.20627736E -2	-.14953007E -2	584
-.18969370E -1	0.29699019E -3	0.10527511E -5	-.73581116E -6	588
0.49464029E -6	0.14950777E -5	-.15086503E -5	-.10342910E -5	592
0.46556625E -6	-.43437891E -6	0.37316912E -6	-.31226761E -6	596
0.25118255E -6	0.16847950E -6	-.17340973E -6	0.17035129E -6	600
-.21205162E -6	-.22554379E -6	0.75005555E -7	-.13581633E -6	604
-.13007377E -7	0.29927470E -7	0.11637070E -6	0.49084788E -7	608
0.73956323E -7	0.78640517E -7	0.46991393E -7	0.67879161E -7	612
0.32622233E -7	-.16941925E -7	-.28068566E -7	0.12936331E -7	616
-.69422689E -4	0.14379517E -3	-.10820297E -2	-.28701095E -3	620
-.15046120E -2	-.12457627E -2	0.14225058E -2	-.14794354E -2	624
0.16955640E -2	-.17113634E -2	0.17213841E -2	-.14277265E -2	628
-.89104268E -2	0.35210225E -2	0.29978378E -3	0.44861419E -5	632
-.38571697E -5	0.13905973E -4	-.20673490E -5	-.11725289E -5	636
0.21471843E -6	-.15414982E -6	0.63714205E -7	-.29311128E -6	640
0.32188784E -6	0.24221382E -6	-.24340351E -6	0.23566197E -6	644
-.23996784E -6	-.21461632E -6	0.87821520E -8	-.20728188E -6	648
0.40009602E -7	-.18464559E -7	0.12101791E -6	0.11297885E -7	652
0.48795206E -7	0.80798124E -7	0.80939485E -7	0.92132232E -7	656
0.69023636E -7	0.43941476E -8	-.14180179E -7	0.38427146E -8	660
0.70825619E -4	-.15004636E -3	0.11034983E -2	0.29934754E -3	664
0.15357319E -2	0.12906560E -2	-.14114630E -2	0.14715719E -2	668
-.17106129E -2	0.17296941E -2	-.17429785E -2	0.13917329E -2	672
0.85161667E -2	-.25136783E -2	0.14964592E -1	0.30024614E -3	676
0.34085566E -5	-.18924309E -4	0.21754510E -5	0.12914653E -5	680
-.33684785E -6	0.27700199E -6	-.18360922E -6	0.30440023E -6	684
-.30967179E -6	-.22734472E -6	0.22951873E -6	-.22284617E -6	688
0.23698021E -6	0.22128180E -6	-.26869101E -7	0.19197242E -6	692
-.26926973E -7	0.63639674E -8	-.12084393E -6	-.20943827E -7	696
-.55511491E -7	-.80729908E -7	-.72883440E -7	-.86566293E -7	700
-.60275455E -7	0.90870265E -9	0.17699685E -7	-.61118469E -8	704
-.69795111E -4	0.15033814E -3	-.10957693E -2	-.29982622E -3	708
-.15135064E -2	-.12860806E -2	0.13610302E -2	-.14217312E -2	712
0.16708642E -2	-.16918369E -2	0.17071718E -2	-.13229126E -2	716
-.79383963E -2	0.17282095E -2	-.13036392E -1	0.11352541E -1	720
0.30248143E -3	0.32528980E -4	-.22135614E -5	-.13574550E -5	724
0.42711427E -6	-.36904577E -6	0.27529821E -6	-.30457782E -6	728
0.29165561E -6	0.20958645E -6	-.21245352E -6	0.20678657E -6	732
-.22805562E -6	-.22019204E -6	0.39962558E -7	-.17485880E -6	736
0.16171871E -7	0.30571662E -8	0.11728274E -6	0.27713709E -7	740
0.59061346E -7	0.78380065E -7	0.64664011E -7	0.79856361E -7	744
0.51887218E -7	-.49250683E -8	-.19877390E -7	0.76660724E -8	748
0.78834228E -4	-.19609320E -3	0.12740035E -2	0.39005654E -3	752
0.17274021E -2	0.16175381E -2	-.12432502E -2	0.13285945E -2	756
-.17579993E -2	0.18053080E -2	-.18469533E -2	0.10146434E -2	760
0.63309693E -2	0.44928084E -2	0.48731822E -1	-.64250172E -1	764
0.10754042E 0	0.29909343E -3	0.25400681E -5	0.20600977E -5	768
-.14289563E -5	0.13929658E -5	-.12966344E -5	0.36989281E -6	772
-.16154202E -6	-.65621972E -7	0.76507150E -7	-.80352907E -7	776
0.18205426E -6	0.25584824E -6	-.18772567E -6	0.31381952E -7	780

0.94033459E -7	-.10353341E -6	-.10673652E -6	-.10580766E -6	784
-.11038547E -6	-.72013387E -7	0.69763956E -8	-.27774492E -7	788
0.24520751E -7	0.48554388E -7	0.47646434E -7	-.25933977E -7	792
-.38685630E -5	0.55660138E -4	-.12612817E -3	-.10911809E -3	796
-.11661538E -3	-.36014160E -3	-.42809544E -3	0.39552089E -3	800
-.12685829E -3	0.84977149E -4	-.42249966E -4	0.72866715E -3	804
0.22396782E -2	-.50288334E -2	-.75138182E -2	0.78742845E -2	808
-.82313009E -2	0.84925575E -2	0.30303310E -3	-.15707178E -4	812
0.19784331E -4	-.16035273E -4	0.13480678E -4	-.74346826E -7	816
-.37433073E -6	-.38117004E -6	0.36334648E -6	-.34000996E -6	820
0.16161969E -6	-.26578368E -7	0.32224528E -6	0.36426401E -6	824
-.24113879E -6	0.21516396E -6	-.49265450E -7	0.16066528E -6	828
0.95926255E -7	-.30881721E -7	-.16500934E -6	-.12724735E -6	832
-.17186419E -6	-.89891294E -7	-.53482420E -7	0.36120844E -7	836
0.26367247E -4	-.23556884E -4	0.36774333E -3	0.48226111E -4	840
0.52728312E -3	0.26466274E -3	-.83843143E -3	0.83931430E -3	844
-.74930357E -3	0.72787469E -3	-.70361570E -3	0.10193634E -2	848
0.33658925E -2	-.37185475E -2	-.47592406E -2	0.52029942E -2	852
-.56551021E -2	0.73280031E -2	-.51833210E -1	0.30429283E -3	856
0.32239477E -4	-.18826308E -4	0.13942159E -4	0.11638137E -6	860
-.42739667E -6	-.38552651E -6	0.37429581E -6	-.35450627E -6	864
0.23881575E -6	0.10027732E -6	0.20810212E -6	0.35230895E -6	868
-.17939018E -6	0.15114814E -6	-.94140320E -7	0.10000640E -6	872
0.38382394E -7	-.60770531E -7	-.14894830E -6	-.12970340E -6	876
-.14741248E -6	-.61251833E -7	-.28139207E -7	0.21843447E -7	880
-.57387236E -4	0.10694429E -3	-.87732895E -3	-.21377873E -3	884
-.11881155E -2	-.91705185E -3	0.12366686E -2	-.12730373E -2	888
0.13767530E -2	-.13779825E -2	0.13743986E -2	-.12826771E -2	892
-.44173683E -2	0.22931143E -2	0.19790672E -2	-.25102528E -2	896
0.30623860E -2	-.61084902E -2	0.70779446E -1	0.10595050E 0	900
0.29910021E -3	0.53971837E -5	-.52236923E -5	-.31881435E -6	904
0.46086381E -6	0.36826171E -6	-.36489578E -6	0.35012585E -6	908
-.30962670E -6	-.23329614E -6	-.71767254E -7	-.31946074E -6	912
0.10257583E -6	-.73443660E -7	0.13778323E -6	-.28773121E -7	916
0.26114170E -7	0.89930841E -7	0.12364073E -6	0.12548105E -6	920
0.11311344E -6	0.27022479E -7	-.77291049E -9	-.52589872E -8	924
0.60740202E -4	-.11704711E -3	0.93390168E -3	0.23379893E -3	928
0.12593149E -2	0.99339542E -3	-.12667621E -2	0.13075728E -2	932
-.14378860E -2	0.14424724E -2	-.14420630E -2	0.12906757E -2	936
0.44306693E -2	-.20050530E -2	-.14986349E -2	0.20195417E -2	940
-.25616106E -2	0.56702868E -2	-.57399977E -1	-.63780896E -1	944
0.18044734E -1	0.29900107E -3	0.51603694E -5	0.34478458E -6	948
-.45846100E -6	-.35946802E -6	0.35735718E -6	-.34359724E -6	952
0.31529140E -6	0.24976114E -6	0.49666731E -7	0.30891915E -6	956
-.88965807E -7	0.60231875E -7	-.14186795E -6	0.17375816E -7	960
-.35454078E -7	-.92655895E -7	-.11742166E -6	-.12243711E -6	964
-.10572142E -6	-.21290235E -7	0.51574579E -8	0.26336418E -8	968
-.63726614E -4	0.12625897E -3	-.98458197E -3	-.25204682E -3	972
-.13226837E -2	-.10625020E -2	0.12911172E -2	-.13359992E -2	976
0.14910680E -2	-.14988156E -2	0.15014109E -2	-.12940023E -2	980
-.44275148E -2	0.17324592E -2	0.10575595E -2	-.15677810E -2	984
0.20993200E -2	-.52533139E -2	0.49087516E -1	0.48769397E -1	988
-.17776118E -1	0.17253699E -1	0.29891655E -3	-.36883340E -6	992
0.45474728E -6	0.34988889E -6	-.34899534E -6	0.33625168E -6	996
-.31979253E -6	-.26487732E -6	-.25223653E -7	-.29781749E -6	1000
0.75619112E -7	-.47359189E -7	0.14534777E -6	-.63821289E -8	1004
0.44306000E -7	0.94979545E -7	0.11109259E -6	0.11913641E -6	1008
0.98305269E -7	0.15731211E -7	-.93420981E -8	-.11128229E -9	1012
0.50752598E -5	0.16474311E -4	0.41634640E -4	-.31720787E -4	1016
0.77390668E -4	-.71051840E -4	-.33232549E -3	0.32066146E -3	1020
-.20692490E -3	0.18739153E -3	-.15700580E -3	0.44710208E -3	1024
0.72999745E -3	-.10351502E -2	-.10611887E -2	0.10977289E -2	1028
-.11309755E -2	0.12164625E -2	-.23314968E -4	0.62912260E -3	1032
-.11086532E -2	0.11744171E -2	-.12339009E -2	0.29917504E -3	1036
-.56579465E -5	-.26956727E -5	0.25219528E -5	-.23126546E -5	1040
0.12486363E -5	0.14152204E -6	0.11044058E -5	0.13673040E -5	1044

-.42053181E -6	0.35680675E -6	-.11906259E -6	0.24357401E -6	1048
0.12521094E -6	-.64983447E -7	-.23890653E -6	-.19269079E -6	1052
-.24027399E -6	-.11024715E -6	-.60582843E -7	0.40119934E -7	1056
0.42414838E -4	-.11781360E -3	0.70065989E -3	0.23331028E -3	1060
0.82054013E -3	0.82658967E -3	-.43825992E -3	0.48167789E -3	1064
-.72735853E -3	0.75582821E -3	-.78194120E -3	0.24310832E -3	1068
0.56078856E -3	0.79423513E -3	0.10850663E -2	-.10298966E -2	1072
0.97095324E -3	-.47021818E -3	-.15802726E -2	-.17273009E -2	1076
0.15745786E -2	-.15373543E -2	0.14979830E -2	-.18911994E -1	1080
0.29912231E -3	0.12885767E -5	-.98993740E -6	0.77674415E -6	1084
0.13048847E -5	0.28167105E -5	-.17000960E -5	-.61015828E -6	1088
0.40873423E -6	-.38933934E -6	-.15702003E -6	-.32482853E -6	1092
-.26633526E -6	-.88974683E -7	0.12135751E -6	0.43877179E -7	1096
0.14933107E -6	0.12219409E -6	0.97424112E -7	-.53627431E -7	1100
0.44388318E -4	-.11535124E -3	0.72221569E -3	0.22864462E -3	1104
0.84885144E -3	0.81668863E -3	-.52699862E -3	0.56773994E -3	1108
-.78621054E -3	0.80984723E -3	-.83098724E -3	0.35833223E -3	1112
0.73126932E -3	0.52748825E -3	0.80382434E -3	-.74097931E -3	1116
0.67528589E -3	-.16596282E -3	-.15305345E -2	-.15180614E -2	1120
0.12569608E -2	-.12052270E -2	0.11529527E -2	-.89288830E -2	1124
0.43078590E -2	0.30043053E -3	0.36807834E -5	-.28916640E -5	1128
0.13156530E -4	0.13345414E -4	-.21292728E -5	-.24153446E -6	1132
0.33417623E -6	-.33076769E -6	-.21540064E -6	-.29071488E -6	1136
-.25992075E -6	-.11759622E -6	0.61781635E -7	-.95069794E -8	1140
0.92015200E -7	0.10043420E -6	0.88018934E -7	-.46183058E -7	1144
-.43112664E -4	0.11321796E -3	-.70308781E -3	-.22437735E -3	1148
-.82504569E -3	-.79952064E -3	0.50099913E -3	-.54119822E -3	1152
0.75877153E -3	-.78259322E -3	0.80391519E -3	-.33200811E -3	1156
-.68503362E -3	-.55242661E -3	-.82199794E -3	0.76211070E -3	1160
-.69932804E -3	0.20645997E -3	0.14887684E -2	0.15000516E -2	1164
-.12631398E -2	0.12150962E -2	-.11662987E -2	0.84764769E -2	1168
-.33622434E -2	0.12249657E -1	0.30120294E -3	0.22599682E -5	1172
-.18003721E -4	-.16050958E -4	0.22384293E -5	0.35280711E -6	1176
-.34954637E -6	0.34263759E -6	0.20568444E -6	0.29725297E -6	1180
0.26105295E -6	0.11225835E -6	-.71579218E -7	0.42563230E -9	1184
-.10120922E -6	-.10336470E -6	-.88944217E -7	0.47000889E -7	1188
0.40780381E -4	-.10783828E -3	0.66608389E -3	0.21369126E -3	1192
0.78072293E -3	0.76017355E -3	-.46701554E -3	0.50543045E -3	1196
-.71459390E -3	0.73767713E -3	-.75839009E -3	0.30391889E -3	1200
0.63182685E -3	0.54739261E -3	0.80289623E -3	-.74700329E -3	1204
0.68829311E -3	-.22353455E -3	-.14084192E -2	-.14337108E -2	1208
0.12202447E -2	-.11762099E -2	0.11313198E -2	-.78305227E -2	1212
0.26634282E -2	-.97153764E -2	0.75031413E -2	0.30388950E -3	1216
0.31335988E -4	0.19899452E -4	-.22547029E -5	-.41580426E -6	1220
0.34738328E -6	-.33847550E -6	-.19270974E -6	-.29123620E -6	1224
-.25286934E -6	-.10510220E -6	0.75156627E -7	0.51910760E -8	1228
0.10338623E -6	0.10158580E -6	0.86427807E -7	-.45866219E -7	1232
-.29184667E -4	0.89158442E -4	-.49322415E -3	-.17630160E -3	1236
-.56689850E -3	-.61005128E -3	0.22616782E -3	-.26019576E -3	1240
0.46522967E -3	-.49061420E -3	0.51453416E -3	-.57014814E -4	1244
-.20851038E -3	-.80561903E -3	-.10066940E -2	0.97720219E -3	1248
-.94431986E -3	0.62011520E -3	0.19685575E -2	0.13228142E -2	1252
-.13318456E -2	0.13214384E -2	-.13080762E -2	0.59611665E -2	1256
0.36981464E -2	0.45518974E -1	-.60546424E -1	0.10311639E 0	1260
0.30033863E -3	-.19223632E -4	0.27789293E -5	0.15074291E -5	1264
-.48262804E -6	0.44060010E -6	0.88813331E -7	0.34435275E -6	1268
0.25520070E -6	0.50003575E -7	-.16967679E -6	-.94307290E -7	1272
-.19140900E -6	-.12853394E -6	-.94036126E -7	0.53279796E -7	1276
-.81256499E -5	0.44271974E -4	-.16417544E -3	-.87018791E -4	1280
-.17130550E -3	-.27714909E -3	-.11203980E -3	0.91950707E -4	1284
0.54950774E -4	-.76414622E -4	0.97822241E -4	0.24406785E -3	1288
0.33486875E -3	-.89404575E -3	-.37969717E -3	0.98833155E -3	1292
-.99379394E -3	0.91614095E -3	0.39835136E -3	0.85056045E -3	1296
-.11134105E -2	0.11444151E -2	-.11708121E -2	0.23937141E -2	1300
0.91083327E -2	0.48616045E -1	-.57656201E -1	0.72082646E -1	1304
-.64006527E -1	0.29907276E -3	0.27507660E -5	0.25221961E -5	1308

-.51839150E -6	0.44766567E -6	-.57345123E -7	0.31799638E -6	1312
0.19176647E -6	-.27920542E -7	-.23699679E -6	-.17481882E -6	1316
-.24537888E -6	-.12684707E -6	-.78564519E -7	0.48145571E -7	1320
-.37312260E -4	0.74873632E -4	-.57647833E -3	-.14908198E -3	1324
-.69145379E -3	-.55926152E -3	0.63421406E -3	-.65602812E -3	1328
0.73630319E -3	-.73965240E -3	0.74045004E -3	-.58705469E -3	1332
-.10119652E -2	0.30801506E -3	0.12118346E -3	-.18882890E -3	1336
0.25625319E -3	-.67029097E -3	0.10751898E -2	0.65976092E -3	1340
-.18166602E -3	0.10561509E -3	-.33247655E -4	0.33224790E -2	1344
-.57263338E -2	-.81176675E -2	0.85778173E -2	-.90365737E -2	1348
0.98413631E -2	0.91976481E -2	0.29944405E -3	-.17418495E -4	1352
0.17579972E -6	-.95639780E -7	0.33903855E -6	0.29651470E -8	1356
0.10606564E -6	0.17040675E -6	0.18379022E -6	0.20070020E -6	1360
0.15645044E -6	0.18700507E -7	-.18195485E -7	0.20422376E -8	1364
-.48062604E -4	0.12059663E -3	-.77589820E -3	-.23912755E -3	1368
-.90716708E -3	-.85133277E -3	0.60109730E -3	-.64221031E -3	1372
0.85576803E -3	-.87774830E -3	0.89687342E -3	-.43721123E -3	1376
-.83158163E -3	-.39544221E -3	-.66427425E -3	0.59703516E -3	1380
-.52745975E -3	0.11650163E -4	0.14915749E -2	0.13986080E -2	1384
-.10864638E -2	0.10277109E -2	-.36927909E -3	0.46460884E -2	1388
-.24567016E -2	-.19413442E -2	0.24698539E -2	-.30185029E -2	1392
0.61304024E -2	0.89675168E -2	-.58072483E -1	0.29903176E -3	1396
-.38105575E -6	0.38418584E -6	0.32169549E -6	0.34386480E -6	1400
0.32089476E -6	0.16287163E -6	-.31638746E -7	0.49661137E -7	1404
-.69433703E -7	-.10337662E -6	-.97091263E -7	0.48609210E -7	1408
0.36849925E -4	-.80964844E -4	0.57867167E -3	0.16082212E -3	1412
0.66879556E -3	0.57383587E -3	-.54178063E -3	0.56594505E -3	1416
-.67275839E -3	0.68064252E -3	-.68622085E -3	0.46109234E -3	1420
0.74192432E -3	-.61643951E -4	0.10002910E -3	-.49230653E -4	1424
-.19019535E -5	0.33060963E -3	-.86923325E -3	-.64387319E -3	1428
0.33949748E -3	-.28870810E -3	0.23991997E -3	-.13333752E -2	1432
0.13766638E -2	0.12971589E -2	-.13681901E -2	0.14352188E -2	1436
-.17158465E -2	-.17272923E -2	0.51207672E -3	-.12742986E -2	1440
0.29896389E -3	-.66058020E -6	-.16798168E -5	-.52733560E -6	1444
-.58312020E -6	-.35100013E -6	-.22860809E -6	-.30206408E -6	1448
-.16010582E -6	0.33773837E -7	0.71048842E -7	-.25033268E -7	1452
-.33136698E -4	0.70593166E -4	-.51728078E -3	-.14029608E -3	1456
-.59851166E -3	-.50303043E -3	0.50449484E -3	-.52498541E -3	1460
0.61101022E -3	-.61650374E -3	0.61999334E -3	-.44027594E -3	1464
-.69587250E -3	0.11563948E -3	-.26806499E -4	-.19619943E -4	1468
0.66133147E -4	-.36379699E -3	0.75986613E -3	0.52911390E -3	1472
-.23874018E -3	0.19118118E -3	-.14572251E -3	0.11196416E -2	1476
-.13047593E -2	-.12023272E -2	0.13175845E -2	-.13688127E -2	1480
0.15542996E -2	0.14844404E -2	-.24311772E -3	0.12819504E -2	1484
-.22095652E -2	0.29896873E -3	0.25422875E -5	0.48674109E -6	1488
0.59925595E -6	0.36870310E -6	0.28855119E -6	0.34853597E -6	1492
0.21580430E -6	-.97417570E -8	-.55645739E -7	0.16617028E -7	1496
0.99199712E -5	-.36577362E -4	0.17615543E -3	0.72092735E -4	1500
0.18893695E -3	0.23183269E -3	-.14251747E -4	0.33379795E -4	1504
-.12635806E -3	0.13879473E -3	-.15085575E -3	-.59477470E -4	1508
-.44401285E -4	0.38281638E -3	0.43059287E -3	-.42631382E -3	1512
0.42062949E -3	-.33960251E -3	-.28056978E -3	-.42643146E -3	1516
0.48627925E -3	-.49028838E -3	0.49271407E -3	-.45212296E -3	1520
-.48926416E -3	-.70623162E -3	0.66323918E -3	-.61764991E -3	1524
0.24286082E -3	-.23411495E -3	0.11977577E -2	0.10577293E -2	1528
-.56000062E -2	0.85035233E -2	0.30004213E -3	-.17416584E -4	1532
-.13545510E -5	-.11022725E -7	0.40819480E -6	0.27399977E -6	1536
0.40220535E -6	0.18574677E -6	0.12008491E -6	-.60833491E -7	1540
-.24544373E -4	0.48078302E -4	-.37730519E -3	-.95703379E -4	1544
-.43859993E -3	-.34869891E -3	0.40718648E -3	-.42006817E -3	1548
0.46504001E -3	-.46611763E -3	0.45557776E -3	-.37535495E -3	1552
-.57257506E -3	0.19708976E -3	0.95586401E -4	-.13165619E -3	1556
0.16740279E -3	-.38399978E -3	0.52998980E -3	0.30609410E -3	1560
-.58205817E -4	0.19168199E -4	0.17751802E -4	0.72393130E -3	1564
-.11107914E -2	-.11271205E -2	0.11558163E -2	-.11807661E -2	1568
0.12159984E -2	0.10338183E -2	0.14749025E -3	0.12069907E -2	1572

-.20853444E -2	0.21216720E -2	-.58047130E -1	0.29894712E -3	1576
0.64265409E -6	0.38740151E -6	0.41085131E -6	0.43480466E -6	1580
0.32902559E -6	0.42005669E -7	-.19418956E -7	-.15114410E -8	1584
-.14661844E -4	0.23468428E -4	-.21812219E -3	-.46931562E -4	1588
-.25748483E -3	-.17980099E -3	0.28626667E -3	-.29119346E -3	1592
0.29513131E -3	-.29210475E -3	0.28803262E -3	-.28685650E -3	1596
-.41883541E -3	0.25405946E -3	0.19616402E -3	-.22030952E -3	1600
0.24380106E -3	-.37234316E -3	0.28887752E -3	0.88183257E -4	1604
0.10368387E -3	-.13226574E -3	0.15888206E -3	0.35889206E -3	1608
-.88037447E -3	-.94431992E -3	0.95087716E -3	-.95422062E -3	1612
0.87271264E -3	0.61870409E -3	0.42251271E -3	0.10702381E -2	1616
-.19668400E -2	0.20382386E -2	-.43897509E -2	0.21497250E -2	1620
0.29896306E -3	0.37117277E -6	0.54824711E -6	0.51496177E -6	1624
0.45445658E -6	0.10274905E -6	0.27246568E -7	-.22205109E -7	1628
0.66813543E -5	-.25436910E -4	0.11971292E -3	0.50100719E -4	1632
0.12590702E -3	0.15797680E -3	-.60388386E -5	0.15768265E -4	1636
-.80528099E -4	0.89266276E -4	-.97766257E -4	-.48568343E -4	1640
-.39857471E -4	0.25884541E -3	0.28670352E -3	-.28436391E -3	1644
0.28109077E -3	-.23112710E -3	-.17361986E -3	-.27206203E -3	1648
0.31499396E -3	-.31808854E -3	0.32011528E -3	-.24173008E -3	1652
-.28302647E -3	-.39097001E -3	0.35788921E -3	-.34345201E -3	1656
0.14500334E -3	-.10944181E -3	0.59998778E -3	0.53873231E -3	1660
-.11663919E -2	0.12282822E -2	0.43935252E -4	0.12932175E -2	1664
0.12415333E -2	0.29905731E -3	0.25579819E -5	0.16560827E -5	1668
0.13856705E -5	0.30726732E -6	0.18682868E -6	-.73589110E -7	1672
0.30428939E -4	-.74399711E -4	0.48800491E -3	0.14742398E -3	1676
0.54468951E -3	0.50215263E -3	-.37086250E -3	0.39392151E -3	1680
-.51138279E -3	0.52270338E -3	-.53229470E -3	0.27448988E -3	1684
0.43661936E -3	0.14164575E -3	0.26029167E -3	-.22752064E -3	1688
0.19394255E -3	0.45287734E -4	-.63867257E -3	-.57174668E -3	1692
0.42006968E -3	-.39184223E -3	0.36411483E -3	-.78244140E -3	1696
0.42446385E -3	0.28972897E -3	-.33408862E -3	0.37747926E -3	1700
-.62870866E -3	-.80117972E -3	0.59949129E -3	-.11768395E -3	1704
-.73846761E -3	0.93629298E -3	0.14479083E -2	0.13593240E -2	1708
0.18232095E -2	0.85534841E -2	0.30025691E -3	-.19270801E -4	1712
0.58291089E -6	0.34137656E -6	0.30998148E -6	-.94568312E -7	1716
0.25913603E -4	-.67530128E -4	0.42134270E -3	0.13367166E -3	1720
0.46658695E -3	0.44930819E -3	-.26122881E -3	0.30298856E -3	1724
-.42077734E -3	0.43326890E -3	-.44435297E -3	0.18392365E -3	1728
0.30963770E -3	0.22370747E -3	0.32525232E -3	-.29958839E -3	1732
0.27289607E -3	-.70705993E -4	-.55778262E -3	-.55204160E -3	1736
0.45761063E -3	-.43781536E -3	0.41791272E -3	-.69397411E -3	1740
0.19280768E -3	0.44427725E -4	-.87901208E -4	0.13124360E -3	1744
-.40721377E -3	-.64882547E -3	0.71268790E -3	0.14818981E -3	1748
-.10371956E -2	0.12047224E -2	0.10978774E -2	0.15308320E -2	1752
0.18326333E -2	0.60866482E -2	-.64181041E -1	0.29905798E -3	1756
0.17935006E -5	0.43158846E -6	0.33770995E -6	-.10874127E -6	1760
0.31669180E -4	-.75271296E -4	0.50489128E -3	0.14921652E -3	1764
0.56428796E -3	0.51029612E -3	-.40274045E -3	0.42556372E -3	1768
-.53827221E -3	0.54853378E -3	-.55696673E -3	0.31028043E -3	1772
0.48200818E -3	0.93019144E -4	0.21475251E -3	-.17986929E -3	1776
0.14433447E -3	0.10259369E -3	-.64872208E -3	-.55370060E -3	1780
0.38017408E -3	-.34900049E -3	0.31861704E -3	-.77578251E -3	1784
0.51234232E -3	0.39254020E -3	-.43418096E -3	0.47448708E -3	1788
-.69557600E -3	-.82098820E -3	0.49793640E -3	-.24133249E -3	1792
-.50809082E -3	0.69026307E -3	0.14014580E -2	0.10795353E -2	1796
0.14998268E -2	0.45803538E -2	0.23262782E -2	0.59971671E -2	1800
0.29899173E -3	0.36331600E -6	0.35369655E -6	-.93886678E -7	1804
0.16857867E -4	-.35361193E -4	0.26222235E -3	0.70249639E -4	1808
0.29501646E -3	0.24528671E -3	-.25067160E -3	0.26030541E -3	1812
-.29986161E -3	0.30206375E -3	-.30323013E -3	0.21807931E -3	1816
0.31511941E -3	-.63429183E -4	-.14563972E -5	0.21682428E -4	1820
-.41866798E -4	0.16961171E -3	-.31431129E -3	-.21148459E -3	1824
0.86548296E -4	-.66226034E -4	0.46880491E -4	-.34397760E -3	1828
0.40944920E -3	0.38189543E -3	-.39577364E -3	0.40833529E -3	1832
-.45198296E -3	-.41862009E -3	0.38365918E -4	-.34705129E -3	1836

0.12148564E -3	-.43751314E -4	0.62375525E -3	0.13337800E -3	1840
0.33586911E -3	0.10033784E -2	0.12267450E -2	0.14358725E -2	1844
0.12151373E -2	0.29895584E -3	0.65023687E -6	-.26715453E -7	1848
0.97714674E -5	-.17902781E -4	0.14841184E -3	0.35666494E -4	1852
0.16907728E -3	0.12863040E -3	-.16623027E -3	0.17049685E -3	1856
-.18248656E -3	0.18200879E -3	-.18089230E -3	0.15647812E -3	1860
0.21841059E -3	-.97501279E -4	-.61840857E -4	0.74780163E -4	1864
-.87493101E -4	0.16185724E -3	-.17247142E -3	-.83525206E -4	1868
-.84689151E -5	0.22596975E -4	-.35851032E -4	-.18065910E -3	1872
0.32275235E -3	0.32356166E -3	-.32833051E -3	0.33198863E -3	1876
-.32390392E -3	-.25492250E -3	-.82666873E -4	-.32393067E -3	1880
0.24253438E -3	-.19340967E -3	0.39307944E -3	-.70660828E -4	1884
0.83904783E -4	0.60142084E -3	0.10994436E -2	0.11190283E -2	1888
0.11803214E -2	0.21750265E -2	0.29896870E -3	0.19853973E -7	1892
-.71295010E -5	0.14064584E -4	-.10964533E -3	-.27969369E -4	1896
-.12292173E -3	-.98122988E -4	0.11173060E -3	-.11529340E -3	1900
0.12808417E -3	-.12839510E -3	0.12826072E -3	-.10076733E -3	1904
-.13963462E -3	0.45626844E -4	0.20572164E -4	-.29111688E -4	1908
0.37564225E -4	-.89081634E -4	0.12240805E -3	0.71749028E -4	1912
-.15497826E -4	0.66207725E -5	0.17620620E -5	0.12355753E -3	1916
-.17904986E -3	-.17249862E -3	0.17656538E -3	-.18003524E -3	1920
0.18598165E -3	0.15844529E -3	0.17341040E -4	0.16274133E -3	1924
-.86131411E -4	0.58728221E -4	-.20203360E -3	-.32376345E -5	1928
-.72232253E -4	-.23967509E -3	-.33752792E -3	-.36167551E -3	1932
-.31398075E -3	-.89506980E -4	0.66408200E -4	0.29897137E -3	1936

Appendice 3

La distortion introduite par l'opérateur lisseur $\mathcal{L} = a_2^2 a_3$

Fréquence no k	Amplitude de l'onde après le filtrage en pourcentage de l'amplitude avant le filtrage %
1	100.0
2	100.0
3	99.9
4	99.9
5	97.6
6	97.6
7	96.8
8	96.8
9	96.8
10	96.7
11	96.7
12	95.8
13	90.6
14	90.0
15	89.3
16	89.2
17	89.1
18	89.1
19	87.6
20	87.5
21	87.5
22	87.4
23	87.4
24	77.9
25	77.1
26	76.0
27	76.0
28	75.9
29	75.8
30	75.7
31	73.9
32	73.6
33	63.3
34	62.0
35	59.7
36	59.4
37	56.9
38	47.4
39	45.0
40	44.8
41	42.5
42	32.9
43	30.8
44	8.4

PREPARATION OF THE DATA

The harmonic method of tidal analysis (using as many as 61 constituents, of which up to 20 are shallow water constituents) gives inaccurate results in the prediction of tides whenever the shallow water effect becomes appreciable. The first step in developing the shallow water analysis was the preparation of the data in a suitable form.

Four computer programs were written to convert, prepare and check the data. Six stations were processed: Québec, Neuville, Grondines, Cap à la Roche, Batiscan and Three Rivers.

Figure 7 illustrates part of the Data Processing System of the Tides, Currents and Water Levels Section. The Shallow Water Analysis and Prediction runs parallel to the Harmonic Analysis and Prediction. Several programs are used by both methods.

Figure 8 shows the flow of data, that is, how the data is processed, prior to the Shallow Water Analysis.

The preparation of the data involved a large amount of data handling and required about twelve man-months. It was divided into five main steps.

1. Placing the data on punched cards:

   Data for 5 stations was available in manuscript form, the data for Québec had to be abstracted from the gauging record. Keypunching (and scaling for Québec) took approximately 6 man-months. Two hundred and fifteen station-years of data or about 1,900,000 digits were processed.

2. Conversion of the card format for high and low waters:

   Because the format of the manuscripts did not agree with the adopted card format, all cards had to be converted with the help of the computer. It took

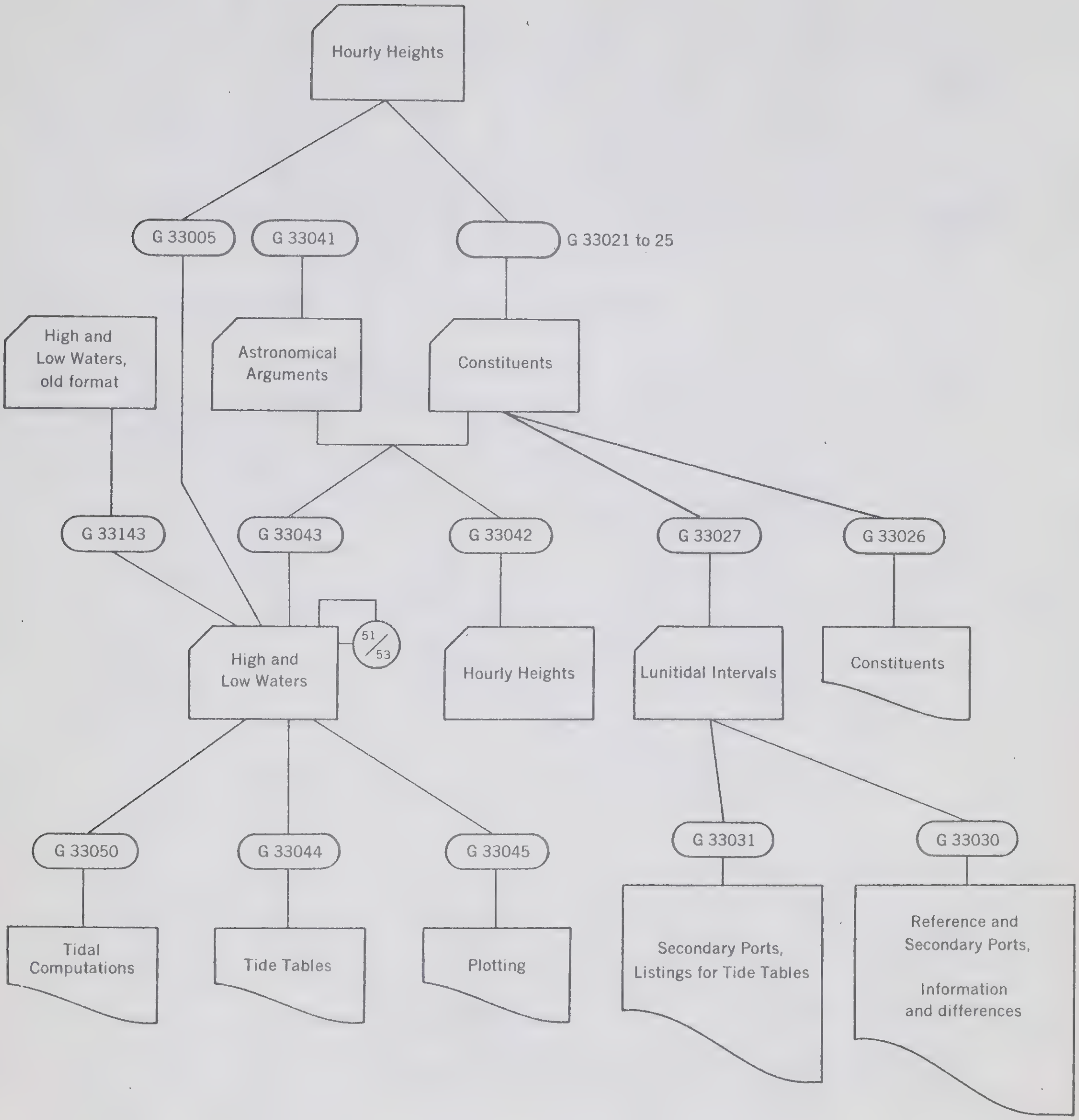


Fig. 7. System of computer programs showing the flow of data for the Harmonic Analysis and Prediction.

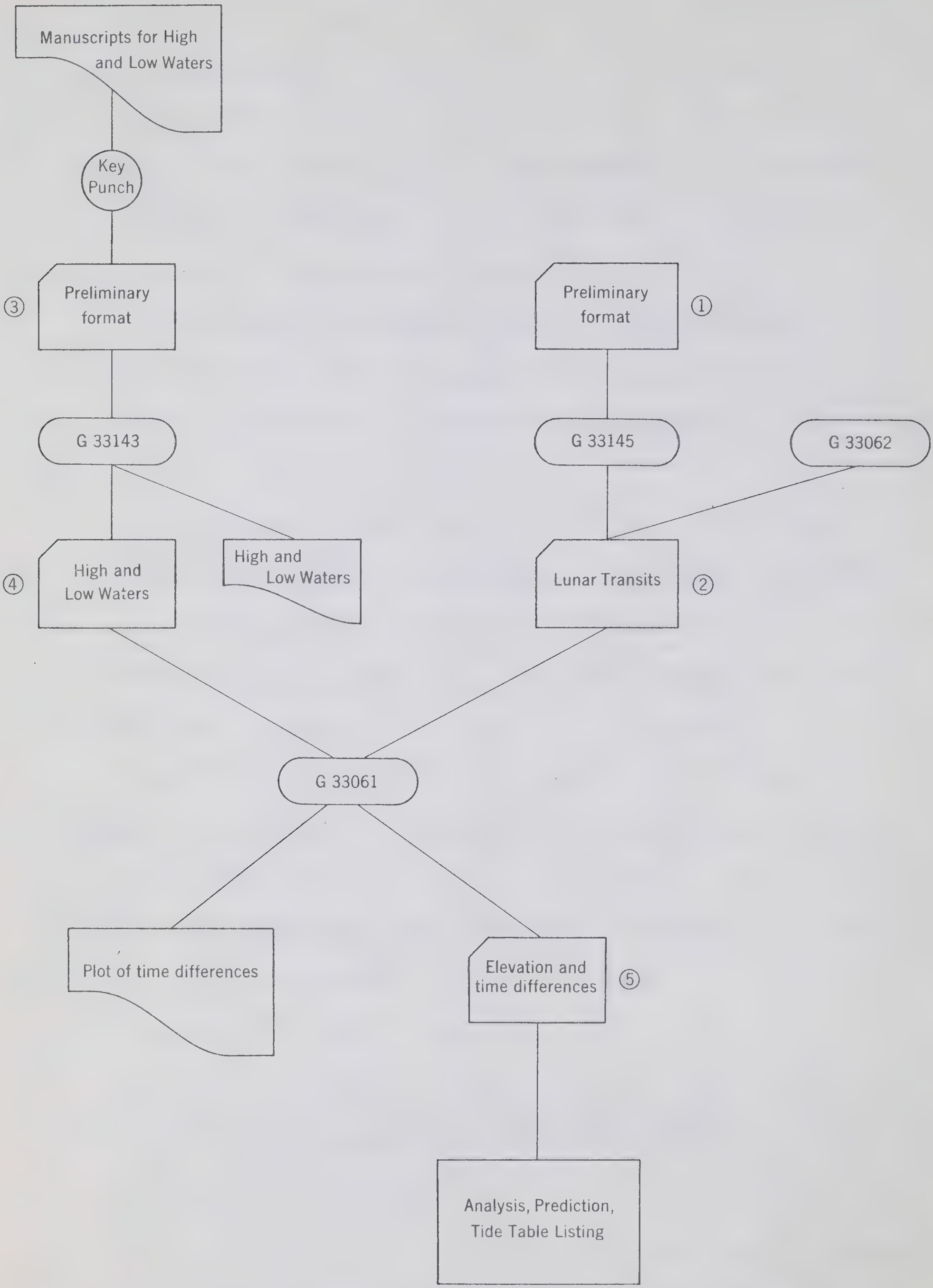


Fig. 8. Flow chart for the preparation of data for the Shallow Water Analysis.

200 man-hours to write the program and 5 hours to process the data on the IBM 1401.

3. Checking and correcting:

Most of the mistakes found in the listing given by the computer could be traced to the manuscripts, even though these had been double checked. Small gaps (up to 8 days) in the record were filled by interpolation. The time intervals (about 24.8 hours) and the height intervals were kept constant in filling a gap. Larger gaps were filled by predicting the intervals using the harmonic method. This part of the work took about 1 man-month.

4. Preparation of lunar transits:

The lunar transits supplied by the German Hydrographic Institute had only one transit per card (see Table 6). A more efficient format was desirable because of frequent card handling. The format designed put 12 lunar transits on one card. A program was written to convert the data.

These lunar transits covered the period of analysis, but the prediction would require lunar transits which were not yet available. Hence a program to produce lunar transits was written, using the IBM 1620. However, it is not economical to run this program on the 1620 so it was converted for the IBM 360.

5. Time differences between the lunar transits and the times of the tide:

Program G33061 computes these time differences. It also gives a printer-plot of the differences, thus enabling one to make a quick and accurate check. The CDC 3100 computer processed the data in about 9 hours. Tables 4 and 5 give the output and the operating instructions for this program. The elevations can be checked with program G33045. Processing and checking took about one man-week per station.

Table 4. Card formats used in Figure 8 in the preparation of data for the Shallow Water Analysis

(1) Lunar transits, preliminary format.

1950110	1046385
195011L	1170719
1950110	2054750
195011L	2179081
1950110	3062927
195011L	3187053
1950110	4070832
195011L	4194598
1950110	5078666
195011L	5202477
1950110	6086284
195011L	6210260

(2) Lunar transits, final format

TU	1150	10438	11704	20529	21754	30618	31842	40705	41928	50752	52015	60838	62102

(3) High and low waters, preliminary format

41411150	104590183	116360210	206020286	217320335	307120266	318580368
41411150	408000315	421000560	509300406	522050233	610280135	622450122
31411150	110001166	122201502	211151177	223201485	312271138	400401314
31411150	413451352	501551593	514401422	603001337	615221326	704051553

(4) High and low waters, final format

91	141	01	1150	0459	1.8	1000	11.7	1636	2.1	2220	15.0
91	141	02	1150	0602	2.9	1115	11.8	1732	3.4	2320	14.9
91	141	03	1150	0712	2.7	1227	11.4	1858	3.7	9999	99.9
90	141	04	1150	0040	13.1	0800	3.2	1345	13.5	2010	5.6
90	141	05	1150	0155	15.9	0930	4.1	1440	14.2	2205	2.3
90	141	06	1150	0300	13.4	1028	1.4	1522	13.3	2245	1.2

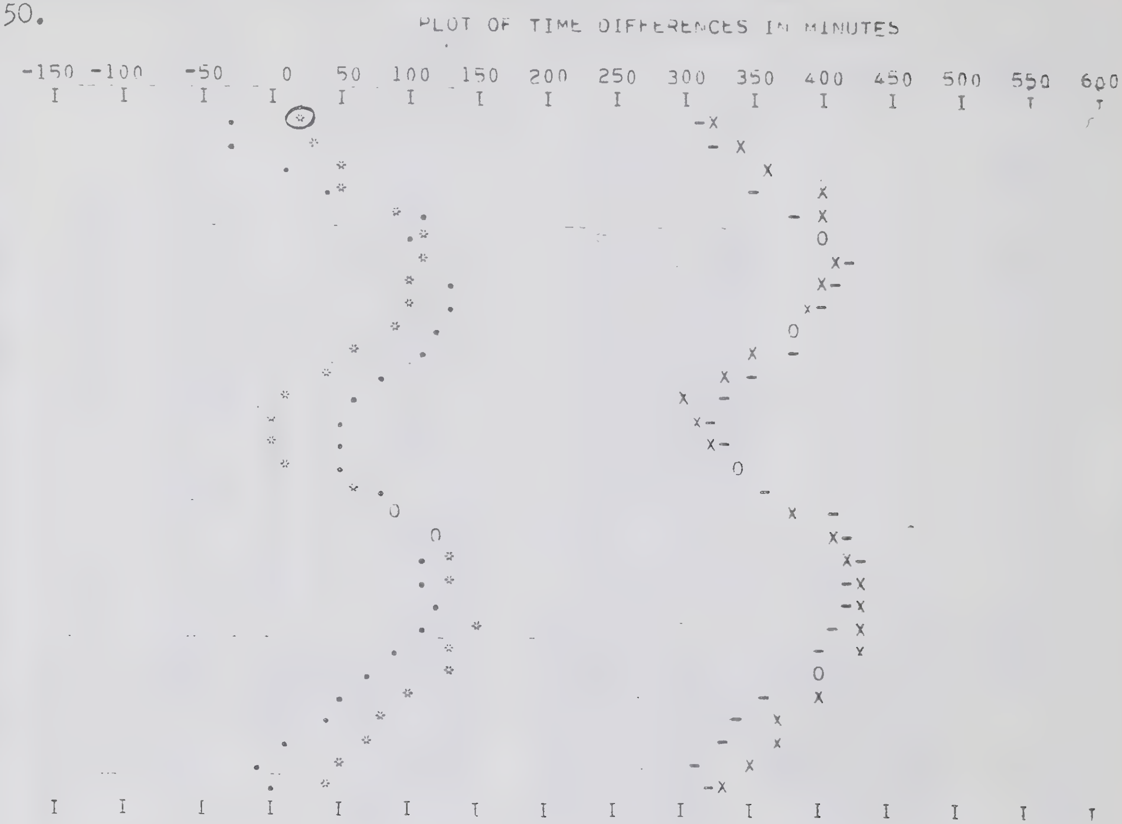
(5) Time differences and elevations

T1	141	1	1150	459	1.8	10 0	11.7	1636	2.1	2220	15.0	21	322	-28	316	2170
T1	141	2	1150	6 2	2.9	1115	11.8	1732	3.4	2320	14.9	33	346	-22	326	2171
T1	141	3	1150	712	2.7	1227	11.4	1858	3.7	9999	99.9	54	369	16	999	2172
T4	141	4	1150	040	13.1	8 0	3.2	1345	13.5	2010	5.6	358	55	400	42	2173
T4	141	5	1150	155	15.9	930	4.1	1440	14.2	22 5	2.3	387	98	408	110	2174
T4	141	6	1150	3 0	13.4	1028	1.4	1522	13.3	2245	1.2	405	110	404	103	2175

Note: The encircled figures are an aid to trace the flow of data. For example: 1/04.6385, the day and hour; 1/04:38, the day and time; 1/04:59, the day and time; and 1/4:59/21, the day, time and time difference.

Table 5. Output of program G33061. Time differences between lunar transits at Greenwich and occurrence of high and low waters at station 1410 for November, 1950.

TIME OF TIDES HRMN	HRMN	HRMN	HRMN	C D	LUNAR TRT HRMN	HRMN	DY	TIME DIFFERENCE MIN	MIN	MIN	MIN
459	10 0	1536	2220	1	438	17 4	1	21	322	-28	316
6 2	1115	1732	2320	1	529	1754	2	33	346	-22	326
712	1227	1858	9999	1	618	1842	3	54	369	16	999
040	8 0	1345	2010	4	7 5	1928	4	358	55	400	42
155	930	1440	22 5	4	752	2015	5	387	98	408	110
3 0	1028	1522	2245	4	838	21 2	6	465	110	404	103
4 5	1120	1623	9999	4	926	2150	7	423	114	417	999
0 6	445	12 5	17 0	3	1016	2243	8	135	415	109	404
055	530	1255	1742	3	1111	2340	9	132	407	104	391
145	6 5	1340	1830	3	1210	9999	10	125	385	90	380
240	7 6	1423	1910	3	042	1314	11	118	384	59	356
315	742	15 5	1952	3	148	1421	12	87	354	44	331
4 2	830	1545	2035	3	254	1527	13	68	336	18	308
455	922	1632	2140	3	358	1628	14	57	324	4	312
556	1035	1724	2250	3	457	1724	15	59	338	0	326
640	1138	1829	2358	3	549	1814	16	51	349	15	344
8 0	1245	20 0	9999	3	637	1859	17	83	358	61	999
120	857	1418	2119	2	720	1941	18	381	97	418	98
235	10 6	15 2	2226	2	8 2	2022	19	414	124	420	124
330	1036	1554	2315	2	842	21 3	20	425	114	432	132
420	1120	1625	2357	2	923	2144	21	437	117	422	133
456	12 5	1710	9999	2	10 5	2227	22	432	120	425	999
058	540	1242	1745	1	1049	2312	23	151	433	113	416
123	622	1311	1621	1	1136	24 0	24	131	430	95	405
213	645	1335	1910	1	1225	9999	25	133	405	70	405
232	732	1410	1922	1	050	1316	26	102	402	54	366
3 4	8 0	1450	1455	1	142	14 8	27	82	378	42	348
345	845	1510	2030	1	234	1459	28	71	371	11	331
417	915	1545	21 2	1	324	1545	29	53	351	-4	313
455	945	1644	22 5	1	413	1637	30	42	332	7	328



PLOTTING SYMBOLS USED FOR THE TIME DIFFERENCE BETWEEN		IF THIS IS THE FIRST FOR THE DAY, USE CODE
LOW WATER AND UPPER TRANSIT	*	1
HIGH WATER AND UPPER TRANSIT	X	2
LOW WATER AND LOWER TRANSIT	.	3
HIGH WATER AND LOWER TRANSIT	-	4

30 DAYS PROCESSED
116 TIDES PROCESSED

<u>Note:</u> The encircled figures are an aid to trace the flow of data. For example, 4:59/1/4:38/21 indicates the day, times and time difference.

Table 6. Specifications for the program G33061 to compute time differences between the lunar transits and the occurrence of the tide.

Input: 2 control cards
lunar transit cards
1 blank card
tidal record cards
1 blank card

The control cards contain: The approximate average of (Low water time - time of lunar transit), the lowest number of the time - difference plot-scale, the amount to be added to the elevations (no decimal point allowed), the card counter decreased by one, the last lunar transit (in minutes) for the day preceding the first tidal record, the page counter decreased by one.

The format of the control cards is: (2 I 4, F 4.1, /, 3 I 4). When starting a job, the field for the counters should be blank. Otherwise, the field on the control card should be equal to that on the last page or card. If the first tidal card is the first for a month, the last lunar transit (in minutes) for the previous day should be entered on the control card, otherwise this field may be left blank.

The lunar transit cards and the tidal cards should be in ascending time sequence. However, if some tidal cards in the middle of a month are mixed up, they will still be processed correctly.

Output: The output consists of a self-explanatory listing and printer-plot, and cards.

On the plot, the symbol 0 indicates the coincidence of two plotting symbols. The cards give the times and the elevations (adjusted, if necessary) of the tide and the time differences between the tide and the lunar transit.

Operator's instruction: Use 11 x 15 paper, one ply.

Put scratch tape on LUN #56.

NOTE: The differences for the first day of record and for the first day after a break in the tidal record should be checked carefully.

THE COMPUTER PROGRAMS

INTRODUCTION

Two computer programs have been written, Program G20607 for the analysis of 19 years of continuous records of time and height of tidal highs and lows, and Program G20618 for the analysis of 19 years of discontinuous records (at least six months of continuous records are needed each summer). Both programs are written in Fortran IV language for the CDC-3100 computer owned by the Department of Energy, Mines and Resources, Ottawa. The running times are 20 minutes for both the continuous and discontinuous programs.

THE TIDAL RECORDS

Nineteen years of daily records occupy anywhere from 4000 to 7000 IBM cards, depending on the size of the gap in the data each winter. For each station being analyzed, the data are checked for sequence and sufficiency using a small auxiliary program (G20618B). If only a few extra days are needed to make sufficient data for a discontinuous analysis, these records are extrapolated by hand and added to the data. Then the data are transferred to magnetic tape (in card image form, using Program 30025 on the IBM 1401 computer), and the tape obtained is used as input to the appropriate analysis program.

COMPARISON OF THE TWO PROGRAMS

The bulk of the computing time used by the analysis programs is taken up in accumulating the right hand sides of the normal equations, which are sums of the form

$$c = \Sigma\ y_j \cos\ j\sigma$$

$$s = \Sigma\ y_j \sin\ j\sigma$$

where y represents an observed time difference or height of an extremum.

The 3100 computer is quite fast at fixed point arithmetic operations, but does not possess the equipment needed to perform floating point operations quickly. To take advantage of this situation in the continuous program, we initially set up fixed point tables of cos $j\sigma$ and sin $j\sigma$, $j = 0 \ldots 59$, for each frequency σ. Using these tables, partial sums of c and s are accumulated successively in fixed point locations. Thus, floating point and trigonometric operations are only required every sixtieth step of the summing process, when the partial sums need to be phased into the grand totals of c and s. In the discontinuous program, the sines and cosines are obtained on successive days using the sin/cos recurrence relations with the initial day analyzed of each year as base. The c and s vectors are accumulated directly in fixed point locations.

The two normal matrices corresponding to the continuous analysis had been pre-computed, triangularized and punched on IBM cards. The continuous program transforms the c and s vectors into least squares coefficients by reading these matrices in a row at a time and performing matrix-vector multiplication. The normal matrices were not available for the discontinuous case, and here the normal equations are generated and solved (by Gaussian Elimination) by the analysis program itself.

The continuous program reads through the data only once, to accumulate c and s. However, the discontinuous program must read it twice, the first time to determine the first and last dates to be used in the analysis each year, and the second time to accumulate c and s.

The detailed operating instructions for the programs are given in the following sections. The decks and listings are on file in the Oceanographic Research Division.

ROUTINES REQUIRED - CONTINUOUS ANALYSIS

1) PROGRAM G20607 (Mainline)
2) SUBROUTINE CDAY
3) SUBROUTINE DMY
4) SUBROUTINE SUB1
5) SUBROUTINE RDMUL
6) FUNCTION ARCTAN

ROUTINES REQUIRED - DISCONTINUOUS ANALYSIS

1) PROGRAM G20618 (Mainline)
2) SUBROUTINE CDAY
3) SUBROUTINE DMY
4) SUBROUTINE SOLVE
5) FUNCTION ARCTAN
6) FUNCTION QUOT

TAPE DATA

The tape data consist of nineteen or more years of daily extrema records in card image form on tape unit 01. There are four or less tidal extremums per day, and the cards are set up as follows:

Column	Entry
1	The letter T
2	The indicator C, a number from 1 to 8
4-7	Station number
9-15	Day, month, year (2 digits)
16-55	Four sets of time (hours and minutes) and height (feet to one decimal) of tidal extremum
56-71	Four sets of time difference (minutes) between the extremum and lunar transit.

FORMAT (A1, I1, 1X, A4, I3, I3, I2, 4(I5, F5. 1), 4I4)

If less than four extrema occur on any day, the remaining fields for time, height and time difference are filled in with 9999, 99.9 and 999 respectively.

The "C" connected with an extremum indicates whether the extremum is a high or a low water and whether the lunar transit used in computing time difference is an upper or lower transit. The "C" on each data card is connected with the first extremum on that card. For successive extrema at any station, C will go through the cycle 12341234. . (or 87658765. . .). C is defined in the tables below.

C	Water	Tran.
1	Low	Upper
2	High	Upper
3	Low	Lower
4	High	Lower

C	Water	Tran.
8	High	Lower
7	Low	Lower
6	High	Upper
5	Low	Upper

CARD DATA

The card data consist of:

1) One card containing the day, month and year on which to start the analysis. FORMAT (I2, 1X, I2, 1X, I2)

2) One card containing s_o, Δs, p_o, Δp, h_o, Δh, N'_o, $\Delta N'$. FORMAT (8F10. 7). The zero subscripted symbols are the values of s, p, h and N' at 00hrs 01/01/1940. The deltas are the rates of increase per 365.00 days.

3) One card for each constituent in the analysis, including the constant term. Each card contains:
   - (i) Constituent number (from 0 to 44)
   - (ii) Frequency σ in cycles per lunar day
   - (iii) The astronomical arguments I1, I2, I3, I4.

   FORMAT (I2, F18. 10, 4X, 4I2)

4) One card with cols 1-2 = 99.

5) These cards are required for the continuous analysis only and consist of:
   - (i) The trangularized 45 X 45 matrix used for determining the cosine least square coefficients, row by row, four elements per card. FORMAT (4E18. 8)
   - (ii) The corresponding 44 X 44 sine matrix in the same format.

CONTINUOUS ANALYSIS - PROGRAMMED DATA CHECKS

1) As each daily observation card is read, its date (cols 9-15) is checked for sequence. If an error is found, the following message is printed and the program continues on.
"ERROR IN SOLAR DAY COUNT, DATE IS XX XX XX."

2) Times of successive lunar transits should differ by about 12 hrs. 25 min. A leeway of 30 minutes is allowed and if an error occurs, the following message is printed and the program continues on.
"ERROR IN LUNAR DAY COUNT, DATE IS XX XX XX."

DISCONTINUOUS ANALYSIS - PROGRAMMED DATA CHECKS

1) The beginning and ending dates of the continuous blocks of data available each year are printed out. This provides a check on the sequence of the data by date, as well as showing how much data is available.

2) As in the continuous case, the lunar transit times are checked. The heights are checked for high-low-high-low... sequence. The indicator "C" on each card is checked for sequence.

   There are three possible error messages here:

SEQUENCE CHECK ON TRANSITS.	C	DD	MM	YY
SEQUENCE CHECK ON HEIGHTS.	C	DD	MM	YY
SEQUENCE CHECK ON C.	C	DD	MM	YY

The symbols "C DD MM YY" represent the values of C and the day month and year where the error occurred. The program continues on after printing an error message.

3) If there is not enough data to perform a discontinuous analysis, the following message is printed and the program stops.
WITH FIRST C = X, ONLY XXX FULL LUNAR DAYS AVAIL.
ANALYSIS NOT POSSIBLE.

PRINTED OUTPUT

The frequency and astronomical arguments connected with each constituent are printed out just as they are read in.

Eight variables are analyzed in these programs; namely height and time difference for each of the four types of extrema (corresponding to C = 1, 2, 3, 4). For each of these variables the amplitude and phase of each constituent is printed out. Along with this are printed out the times of the two lunar transits on the middle day of analysis, and "T", "C" and station number picked off the first daily record used in the analysis.

In the discontinuous analysis, tables giving the first and last dates of the data available, and the first and last dates used each year are also printed out.

Table 7 shows a sample printout from the discontinuous analysis.

PUNCHED CARD OUTPUT

1) The punched card output consists of one card containing the following:

Col.	Format	Entry	
1	A1	The letter "T"	
2	I1	The indicator C from first data card	
4-7	A4	Station number	
11-12	I2	Hours	
13-14	I2	Minutes	Time of First Lunar
21-22	I2	Day	Transit on Middle
23-24	I2	Month	Lunar Day.
25-26	I2	Year	
31-32	I2	Hours	
33-34	I2	Minutes	Time of Second Lunar
41-42	I2	Day	Transit on Middle
43-44	I2	Month	Lunar Day.
45-46	I2	Year	

2) A pair of cards for each constituent analyzed.

The first card of each pair contains the constituent number, the four pairs of amplitude and phase lag corresponding to the left half of the printed output, and a "1" in col. 80.

FORMAT (I2, F8.1, F6.1, F8.2, F6.1, F10.1, F6.1, F8.2, F6.1, 18X, 12)

The second card of each pair contains the constituent number, the eight values from the righthalf of the printed output and the number "2" in col. 80. The format is identical.

Table 7. A sample output in four parts of the 19 year discontinuous analysis.

19 YEAR DISCONTINUOUS ANALYSIS

S0	DS/YEAR	H0	DH/YEAR	P0	DP/YEAR	N0	DN/YEAR
.4750300	13.3594023	.7764200	.9993369	.4496700	.1129513	.4290300	.0536894

NO.	SIGMA	I1	I2	I3	I4
0	0	0	0	0	0
3	.0050271437	0	2	-2	0
4	.0056677466	0	2	0	0
5	.0322162254	1	-2	0	0
6	.0325365285	1	-2	1	0
7	.0374114225	1	0	-1	-1
8	.0375636722	1	0	-1	0
9	.0377317239	1	0	0	-1
10	.0378839737	1	0	0	0
11	.0380362234	1	0	0	1
12	.0432314188	1	2	-1	0
13	.0650730570	2	-4	2	0
14	.0672663275	2	-3	0	0
15	.0699479510	2	-2	0	-1
16	.0701002007	2	-2	0	0
17	.0702524505	2	-2	0	1
18	.0704205021	2	-2	1	0
19	.0751273445	2	0	-2	0
20	.0754476459	2	0	-1	0
21	.0756156976	2	0	0	-1
22	.0757679473	2	0	0	0
23	.0759200304	2	0	0	1
24	.1026367292	3	-4	1	0
25	.1048299997	3	-3	-1	0
26	.1075116232	3	-2	-1	-1
27	.1076638729	3	-2	-1	0
28	.1078161227	3	-2	-1	1
29	.1079841744	3	-2	0	0
30	.1083044758	3	-2	1	0
31	.1126910167	3	0	-3	0
32	.1133316195	3	0	-1	0
33	.1373665282	4	-5	0	0
34	.1402004014	4	-4	0	0
35	.1452275452	4	-2	-2	0
36	.1458681480	4	-2	0	0
37	.1515358946	4	0	0	0
38	.1727369300	5	-6	1	0
39	.1777640737	5	-4	-1	0
40	.1780843751	5	-4	0	0
41	.1834318202	5	-2	-1	0
42	.2103006022	6	-6	0	0
43	.2159683487	6	-4	0	0
44	.2804008030	8	-8	0	0

Table 7 (cont.)

19 YEAR DISCONTINUOUS ANALYSIS T 1290 BLOCKS OF CONTINUOUS DATA AVAILABLE

L.YR	C	DATE		C	DATE	TRANSIT	TIMES(LUN.DYS)		DIFF
1	1	3/ 5/35	TO	3	12/11/35	0	TO	186.51	186.51
2	1	1/ 5/36	TO	2	12/11/36	-0.99	TO	187.99	188.98
3	3	26/ 4/37	TO	4	17/11/37	-4.51	TO	193.48	197.99
4	1	26/ 4/38	TO	1	16/11/38	-4.02	TO	193.01	197.03
5	3	5/ 5/39	TO	2	15/11/39	5.51	TO	193.01	187.51
6	3	1/ 5/40	TO	2	18/11/40	2.51	TO	196.98	194.47
7	1	24/ 4/41	TO	2	15/11/41	-3.00	TO	194.98	197.98
8	1	1/ 5/42	TO	2	19/11/42	3.99	TO	199.01	195.03
9	3	5/ 5/43	TO	4	15/11/43	8.51	TO	196.51	188.00
10	4	29/ 4/44	TO	4	17/11/44	4.52	TO	199.48	194.97
11	3	20/ 4/45	TO	3	15/11/45	-3.48	TO	198.49	201.96
12	3	22/ 4/46	TO	2	15/11/46	-0.51	TO	199.03	199.54
13	3	28/ 4/47	TO	2	15/11/47	5.52	TO	200.00	194.49
14	3	21/ 4/48	TO	2	17/11/48	.51	TO	202.98	202.47
15	4	21/ 4/49	TO	2	15/11/49	.52	TO	201.99	201.46
16	4	26/ 4/50	TO	3	16/11/50	6.49	TO	203.53	197.04
17	3	21/ 4/51	TO	4	15/11/51	2.48	TO	203.50	201.02
18	1	22/ 4/52	TO	2	15/11/52	5.00	TO	204.97	199.97
19	1	20/ 4/53	TO	1	15/11/53	4.02	TO	205.99	201.97

BLOCKS OF DATA USED EACH YEAR

1	1224	12/ 5/35	TO	848	11/11/35
2	1950	10/ 5/36	TO	1740	9/11/36
3	407	10/ 5/37	TO	2330	8/11/37
4	1313	9/ 5/38	TO	833	8/11/38
5	2046	8/ 5/39	TO	1710	7/11/39
6	440	7/ 5/40	TO	57	6/11/40
7	1208	6/ 5/41	TO	917	5/11/41
8	2115	5/ 5/42	TO	1736	4/11/42
9	442	5/ 5/43	TO	22	4/11/43
10	1358	3/ 5/44	TO	930	2/11/44
11	2040	2/ 5/45	TO	1900	1/11/45
12	512	2/ 5/46	TO	28	1/11/46
13	1421	1/ 5/47	TO	925	31/10/47
14	2135	29/ 4/48	TO	1836	29/10/48
15	524	29/ 4/49	TO	148	29/10/49
16	1342	28/ 4/50	TO	1003	28/10/50
17	2204	27/ 4/51	TO	1826	27/10/51
18	549	26/ 4/52	TO	116	26/10/52
19	1507	25/ 4/53	TO	1010	25/10/53

Table 7 (cont.)

ANALYSIS OF 19 YEARS OF DISCONTIN. OBSERVATIONS
ON THE TIME AND THE HEIGHT OF HIGH AND LOW WATER

10H 5M 2/ 8/44 T 3 1290 224 35M 2/ 8/44

	LOWER TRANSIT								UPPER TRANSIT								
	LOW WATER				HIGH WATER				LOW WATER				HIGH WATER				
	TIME MIN	DIFF DEG	HGT FT	DIFF DEG	TIME MIN	DIFF DEG	HGT FT	DIFF DEG	TIME MIN	DIFF DEG	HGT FT	DIFF DEG	TIME MIN	DIFF DEG	HGT FT	DIFF DEG	
0	270.8	0	3.38	0	508.0	0	10.15	0	271.0	0	3.37	0	508.0	0	10.14	0	0
3	1.3	21.5	.22	303.6	.6	106.6	.14	309.8	1.0	24.7	.22	303.1	.8	105.0	.14	310.3	3
4	7.9	137.3	1.59	137.5	5.4	334.2	1.02	140.5	7.7	136.1	1.58	138.1	5.7	330.9	1.01	141.7	4
5	1.8	44.2	.06	35.8	1.5	180.4	.08	258.6	1.9	231.6	.03	218.6	1.7	341.3	.12	74.3	5
6	8.3	119.0	.03	291.7	7.9	111.8	.08	194.2	8.3	120.0	.03	286.8	7.8	111.1	.07	195.2	6
7	.5	352.9	.04	62.8	.4	293.9	.03	75.1	.1	315.7	.03	76.2	.0	317.4	.03	38.5	7
8	9.8	167.1	.22	61.8	11.9	185.4	.67	30.9	9.7	167.5	.22	59.0	12.2	184.0	.66	31.2	8
9	.6	347.0	.04	23.8	.6	333.3	.03	84.5	.6	93.9	.04	21.2	.5	146.0	.02	20.0	9
10	25.3	278.3	.09	269.7	11.0	282.7	.93	97.9	21.9	102.4	.11	116.1	8.2	109.7	.94	275.6	10
11	4.6	276.3	.01	206.8	2.1	256.1	.16	103.8	3.8	107.8	.04	161.3	2.0	89.6	.18	276.0	11
12	.4	38.6	.04	253.1	.7	6.8	.06	206.2	.2	129.2	.04	269.6	.2	101.5	.05	213.3	12
13	.8	128.6	.01	252.9	.5	111.6	.02	271.9	.4	101.9	.01	285.4	.5	95.0	.01	185.9	13
14	1.4	6.5	.11	204.3	1.4	2.7	.05	84.4	1.6	2.0	.10	213.0	1.1	14.1	.05	77.5	14
15	1.4	311.1	.00	10.6	1.6	294.0	.05	35.2	1.3	336.6	.01	10.3	1.3	299.7	.04	36.4	15
16	34.9	317.6	.50	83.3	38.3	301.3	1.22	42.0	35.1	317.2	.48	83.3	38.0	301.9	1.22	42.0	16
17	.2	10.9	.03	243.0	.1	88.9	.03	249.3	.4	269.4	.03	239.1	.3	345.0	.03	276.4	17
18	1.3	285.9	.02	269.0	.7	284.4	.02	116.1	.9	109.1	.03	313.8	1.3	148.5	.04	269.5	18
19	.8	213.1	.01	26.4	.8	211.6	.04	31.8	.6	181.1	.01	57.0	.5	232.1	.04	357.5	19
20	1.2	295.4	.02	221.2	.2	322.0	.07	127.7	1.2	123.9	.02	178.0	.1	63.5	.05	274.4	20
21	.3	40.2	.02	83.8	.2	288.5	.03	55.4	.4	31.3	.01	77.7	.2	18.8	.02	87.5	21
22	8.9	20.2	.10	58.3	6.1	358.5	.28	39.5	8.6	19.3	.08	49.4	6.0	2.1	.31	39.0	22
23	3.3	41.2	.03	58.2	2.2	43.0	.10	40.5	3.1	44.4	.03	67.5	2.1	42.0	.11	30.1	23
24	2.7	335.8	.01	115.9	2.4	316.8	.06	70.1	2.4	335.8	.02	137.9	2.4	319.7	.06	62.1	24
25	.3	162.0	.00	166.9	.4	103.8	.01	78.2	.5	99.8	.02	168.4	.6	28.7	.01	133.9	25
26	.3	20.3	.01	299.7	.1	8.9	.02	292.0	.2	68.4	.02	314.2	.2	4.4	.01	324.7	26
27	1.5	352.7	.04	44.9	1.7	341.9	.07	36.1	1.7	354.5	.05	58.7	1.9	330.1	.08	47.7	27
28	.4	260.8	.02	20.3	.3	307.8	.03	36.2	.4	103.0	.01	40.8	.1	224.5	.02	342.1	28
29	3.5	127.1	.07	122.3	2.3	145.8	.03	275.2	2.4	323.9	.10	169.5	2.9	349.0	.12	132.2	29
30	1.2	338.3	.01	95.2	1.2	319.3	.01	34.0	1.0	325.0	.01	83.9	1.1	312.9	.03	29.7	30
31	.2	103.6	.01	292.8	.3	96.5	.01	356.3	.2	16.2	.00	344.1	.0	290.8	.00	202.2	31
32	1.5	51.0	.03	50.0	.9	61.6	.05	51.2	1.5	47.9	.02	70.8	1.2	36.3	.04	45.7	32
33	1.0	227.1	.01	218.7	.6	211.0	.02	314.1	.5	237.0	.01	30.8	.3	252.2	.03	255.9	33
34	5.7	176.2	.04	341.5	5.9	163.9	.14	254.3	5.5	175.9	.04	332.7	5.4	162.9	.13	258.8	34
35	.1	312.6	.00	36.6	.2	247.4	.02	71.5	.3	28.3	.01	103.8	.1	42.5	.01	140.2	35
36	2.9	171.4	.02	302.0	3.4	155.2	.07	250.2	3.0	174.0	.02	295.7	3.0	153.1	.07	250.3	36
37	.4	195.1	.01	297.2	.5	178.4	.02	241.0	.3	225.0	.02	289.0	.4	198.4	.02	239.7	37
38	.7	169.6	.01	33.1	.9	158.7	.01	309.0	.8	181.9	.02	60.7	.7	194.2	.01	285.4	38
39	.4	215.2	.01	346.6	.4	208.3	.01	305.3	.8	178.6	.01	354.3	.7	189.2	.02	305.7	39
40	.6	323.6	.01	232.2	.6	357.1	.01	211.1	.7	191.3	.01	347.1	.9	186.2	.02	327.0	40
41	.3	222.1	.01	294.2	.4	140.3	.01	227.8	.4	176.2	.01	284.2	.5	164.8	.02	250.8	41
42	1.1	29.2	.02	126.5	1.4	2.8	.03	93.2	1.1	36.1	.02	137.7	1.0	12.4	.03	124.5	42
43	.8	25.1	.03	201.3	1.1	11.0	.02	138.9	.7	31.0	.02	235.3	.8	19.7	.01	140.2	43
44	.2	222.3	.01	346.8	.5	195.8	.01	260.4	.2	276.2	.01	252.3	.3	214.0	.02	289.9	44

Manuscript Report Series No. 4

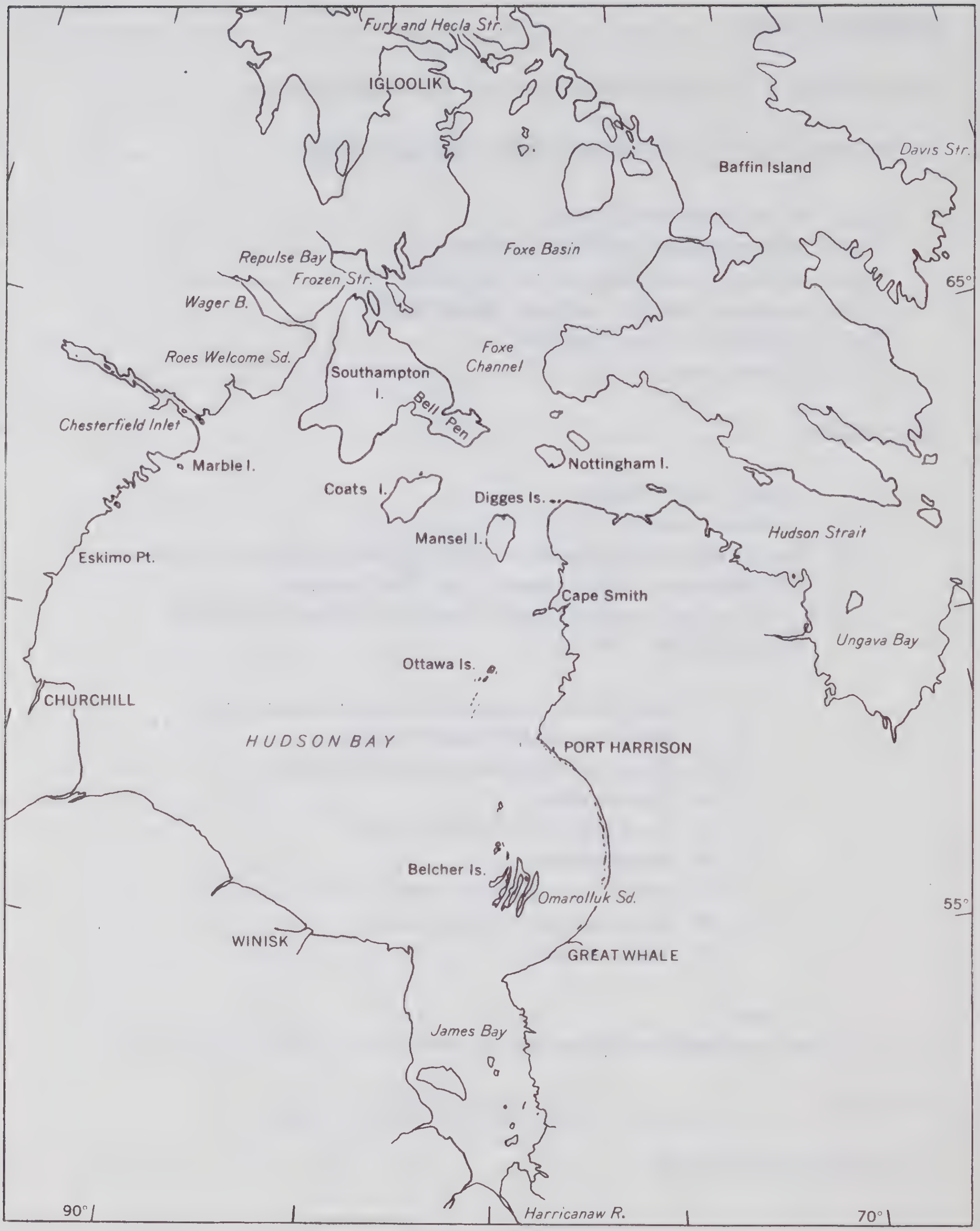
A CONTRIBUTION TO THE OCEANOGRAPHY OF HUDSON BAY

F. G. Barber

1967

ROGER DUHAMEL, F.R.S.C.
QUEEN'S PRINTER AND CONTROLLER OF STATIONERY
OTTAWA, 1967

CONTENTS



Place names in the Hudson Bay region.

1. INTRODUCTION

It is the intention to provide in this work a description of the distribution of certain of the physical features and properties of the water of Hudson Bay, and to remark on the more important processes which likely determine these distributions. The region is still only imperfectly known so that much of the discussion regarding processes is based on experience and understanding gained in study of other regions, both coastal and oceanic. It is to be expected however that Hudson Bay would exhibit features quite unique, for it is notable not only for a general shallowness relative to the total area, but also for the continental location, the existence of a winter ice cover, and the marked amplification, in certain areas, of the oceanic tide. The integrated effect of these and other conditions leads to the observed features out of which understanding evolves. At present this is mainly qualitative and, in some important aspects, also speculative.

As is apparent from the description in later sections, differences of density under most conditions occurring in Hudson Bay are determined by differences of salinity; temperature plays a secondary role as the range is relatively small. Also the lower limit of the range is frequently close to the freezing point. This provides a partial filter with respect to certain data in that their correctness is suspect if they are much below the freezing point. An example is the temperature data reported by Hachey (1931a), which include a number of unrealistically low values.

Hachey (1931a) provided a summary of the expeditions and workers in Hudson Bay prior to the 1930 Hudson Bay Fisheries Expedition in "Loubyrne" (Hachey, 1931b), and remarked therein, "the first known attempt to determine the water temperatures and densities at the various depths was made by Beauchemin in 1929". In this, stations were occupied in Hudson Strait, in Hudson Bay along the route to the port of Churchill, and in Churchill Harbour. A generally similar series of observations was made in the navigation season of each of the years 1930 and 1931; the data for the three years have been reported (Anon., 1931a; 1931b; 1932), and the vessel in which the surveys were conducted, the "Acadia", has been described (Anon., 1964c).

Examination of the above reports and the relevant field books, as well as discussions with Mr. J. A. Deveault, who was a member of the "Acadia" survey staff during each of the three seasons and who participated in the sampling program, have indicated that two protected thermometers were attached to each reversing bottle, and that a cast consisted of a lowering of one bottle.

> "---- the ship was equipped with the latest pattern of Nansen Stopcock reversing deep-sea water bottles of 1 1/2 litre capacity, each bottle being equipped with two Negretti and Zambra reversing deep-sea thermometers, registering from -2 to 25° Centigrade: each thermometer bearing a certificate from the National Physical Testing Laboratory. Negretti and Zambra hydrometers were used to obtain the densities, the usual practice at 60° Fahrenheit being closely followed." (from Anon., 1931a, p. 112).

Table I. A tabulation of the available material, much of which was examined in the preparation of this work, indicating the year obtained, name of ship, Canadian Reference Number, and at least one reference. Most of the material is available in the records of the Canadian Oceanographic Data Centre and is listed in the CODC index (Anon., 1965, a). An asterisk in the last column indicates that the publication is a data report or contains data in tabular form.

Year	Ship	CRN	Reference
1929	"Acadia"	24	Anon., 1931, a*
1930	"Loubyrne"	29	Hachey, 1931, a*
1930	"Acadia"	32	Anon., 1931, b*
1931	do	35	Anon., 1932*
1948	"Haida"	100	Bailey et al. 1951 and Dunbar, 1958*
1953	"Calanus"	193	Grainger et al. 1956
1954	do	186	do
1955	"Labrador"	203	Campbell, 1959
1956	do	211	do
1956	do	219	do
1955	"Calanus"	---	Grainger et al. 1959
1958	do	---	do
1958	do	320	Grainger, 1960*
1959	do	321	do
1959	"Labrador"	323	
1961	"Calanus"	354	Anon., 1964, b*
1961	"Theta"	337	Anon., 1964, a*
1962	"John A. MacDonald"	359	Anon., 1966*

In the available records only the corrected value for each thermometer is shown so that it was not possible to check the corrections to the temperature data, but the material does appear to be free of values which, at the present level of our understanding of the region, might be considered anomalous*. The distribution of temperature at 75m depth as obtained from these observations is shown in a figure and discussed here in the appropriate section; the density values have not been utilized. Also it appears that two samples were drawn from each water bottle and carefully stored for analysis ashore which was to include some form of chemical determination. No record of this has been found.

With regard to the work of "Acadia" mention should be made of a program of drift-bottle releases during the autumn of 1929 on the return voyage between Churchill and eastern Hudson Strait (Anon., 1931a). Eighteen bottles were released but there is no record of any returns (Dr. H. B. Hachey, personal communication).

The first determination of the salinity of the water of Hudson Bay was that reported by Hachey (1931a) in a discussion of the "Loubyrne" data. This indicated a relatively low level of salinity generally, which in part led Bailey et al. (1951) to a comparison with the 1948 "Haida" observations and to conjecture concerning an increase in the influence of Atlantic water in Hudson Bay. The data utilized in the present work include those obtained in the vessels mentioned above as well as those obtained in the vessels listed in Table I. With regard to sub-surface temperature and salinity data obtained through shipborne survey, it is believed that the listing is complete.

A useful group of observations in Hudson Bay is that carried out in "Calanus"; the extent of these observations to 1958 has been described in three station lists (Grainger, 1954; Grainger et al., 1956; 1959), and a summary of the available results to 1955 was provided by Dunbar (1956). Extensive observations in more northerly portions of the system including Hudson Strait were made in 1955, 1956, and 1959 in the icebreaker "Labrador". Part of this material has been utilized in a number of reports including those of Campbell et al. (1956; 1958) and Campbell (1958; 1959; 1964).

Most of the observations described here were made in 1961 in the motor vessels "Calanus" and "Theta" and in 1962 in the Department of Transport icebreaker "John A. MacDonald". The program of "Calanus" and "Theta" included observations for biological, geological, and geophysical as well as physical oceanographic observations. The latter have been reported in data record form (Anon., 1964a; b), atlas form (Barber et al., 1964), and discussions of most of the other observations have been given by Grainger (1963), Hood (1964; 1966), Leslie (1963; 1964; 1965), Leslie et al. (1965), and Barber (in review). The survey of 1962 comprised one of the most wide-ranging shipborne surveys conducted in the Canadian Arctic in one season. Stations were occupied from McClure Strait on the west to Nansen Sound in the north

---

*The cold water (to -1.8°C) to 541m noted at "Acadia" station 19 on October 30, 1929 in eastern Hudson Strait is of interest as no other such occurrence has been recorded although a number of stations have been occupied since.

as well as in Hudson Strait and in Hudson Bay where a number of the positions occupied in 1961 were re-occupied. The material has been reported (Anon., 1966).

The general paucity of observations at times other than the navigation season is to be recognized and comprises a serious limitation to study of the region. An attempt to predict the probable winter surface salinity within the Hudson Bay-Foxe Basin system is made here in the discussion where the few known observations which have been made from ice cover are utilized. Elsewhere in this work similar attempts to extend available data are made in order that frequently used methods of oceanography might be applied, occasionally to obtain a quantitative result. This is sufficient reason for their inclusion, but it may be that Hudson Bay provides a situation similar to the Arctic Sea in that it is particularly attractive to the oceanographer for such calculation and deduction (Pickard, 1964, p. 152). The limits imposed on this by the author are believed to have been realistic.

It may also be mentioned that the author's "local" knowledge is not intimate, and is based mainly on the experience gained on the survey of 1961 in "Theta". Usually this need not be more than a minor hindrance, but as Hudson Bay remains less than well-known there exists a significant probability that an important feature could be overlooked. Information and advice in publications, including charts and Pilots of the Canadian Hydrographic Service reduced this considerably and, in addition, provided background material essential for the work.

2. DESCRIPTION OF THE DISTRIBUTION OF PROPERTIES

A pictorial presentation or atlas of the distribution of temperature, salinity, and dissolved oxygen observed in 1961 has been compiled by Barber et al. (1964). As apparent there, and as described later here, the manner in which the ice cover dissipates each season can have a marked influence on near surface property levels, particularly those of temperature and salinity. The distributions of these properties receive emphasis here although the distribution of sigma-t*, as derived from salinity and temperature data, and the dissolved oxygen distribution are described also. The vertical distribution or structure of sigma-t follows closely that of the salinity as the temperature range over the depth is small, particularly away from the surface mixed layer.

The probable evolution of the mixed layer in Hudson Bay is described in a later section and Barber (in review) provided information concerning the distribution

---

*Sigma-t may be considered a specific gravity anomaly (Fofonoff, 1962, p. 9) such that

$$\text{sigma-t} = 10^3 \, (\rho_{sto} - 1)$$

where ρ_{sto} is the value of the specific gravity calculated for the water sample at the observed or "in situ" salinity, s, and temperature, t, and at atmospheric pressure (see also Montgomery, 1938, p. 11).

of the layer and the associated thermocline as observed in August, 1961. At that time a well-defined layer occurred over most of the region; in the central portion depths up to 20m and thermoclines with intensities in excess of 6C°/25m were observed. Bathythermograph observations during the period September 14 to 29, 1962 in "John A. MacDonald" indicate the existence across the central region of layer depths to 30m with temperatures to 5°C and very well-defined thermoclines. The layer depth and thermocline structure then becomes a marked feature of the late summer and early autumn temperature distribution in the near surface in the absence of ice. Earlier in summer, in July, surface temperatures (Fig. 1, a) in some areas were relatively high, up to 11°C, while close to an ice cover values close to the freezing point were observed. Other features of the layer depth and thermocline at this time are indicated in Figure 1, b, c (a re-presentation of these July data is made later in terms of seasonal heat storage). The thermocline frequently extended to the surface, so that a layer depth did not exist or was shallow or ill-defined. In the north central region depths to 10m are indicated.

With the exception of the areas of strong tidal mixing the features associated with the seasonal thermocline, the surface temperature, layer depth, and thermocline intensity, directly reflect the amount of heat absorbed at the surface during the heating of summer. In Hudson Bay this is largely determined by the duration of the ice cover of the previous winter. Accompanying the absorption is a downward mixing of the heat which results in the generally observed layer depth in which the water is isothermal and isohaline. The downward mixing can be inhibited through the existence in the immediate surface of low salinity water. This frequently occurs close to coasts and in sheltered areas due to a local inflow of fresh water. The amount of heat absorbed may be similar to that in other areas but as its depth distribution is less, relatively high temperatures are observed at the surface.

Here, and generally in August (Fig. 2, a) salinity values over most of the area were relatively low, ranging from less than 10°/00 in southern James Bay to 30°/00 in about latitude 61°N. Values to 32°/00 were observed north of this latitude from Roes Welcome Sound to Hudson Strait, where a high of 32.5°/00 was observed south of Nottingham Island. The gradient at the surface from north to south was generally similar throughout, except south and east of the Belcher Islands where a relatively steep gradient is indicated. This likely was caused by local inflow of fresh water in the region from Cape Jones to Richmond Gulf where a surface value as low as 10°/00 has been observed. Steep gradients likely occur as well in the vicinity of a melting ice cover, as suggested in the figure for the area seaward of Churchill. This was probably more pronounced in July (see inset i to Figure 2, a) at which time values in excess of 32°/00 were observed to the north of Churchill as well as in Roes Welcome Sound. At this time in the vicinity of Chesterfield Inlet surface salinities were lower than in August, suggesting the influence of fresh water from run-off from the inlet where a value of 21°/00 was observed.

The data for August and September, 1930 described by Hachey (1931, b) indicate a similar overall north-south gradient, although there is a tendency toward a northwest-southeast gradient rather than north-south. Values in the north and northwest for that time appear to be less than those indicated here in Figure 2, a. These

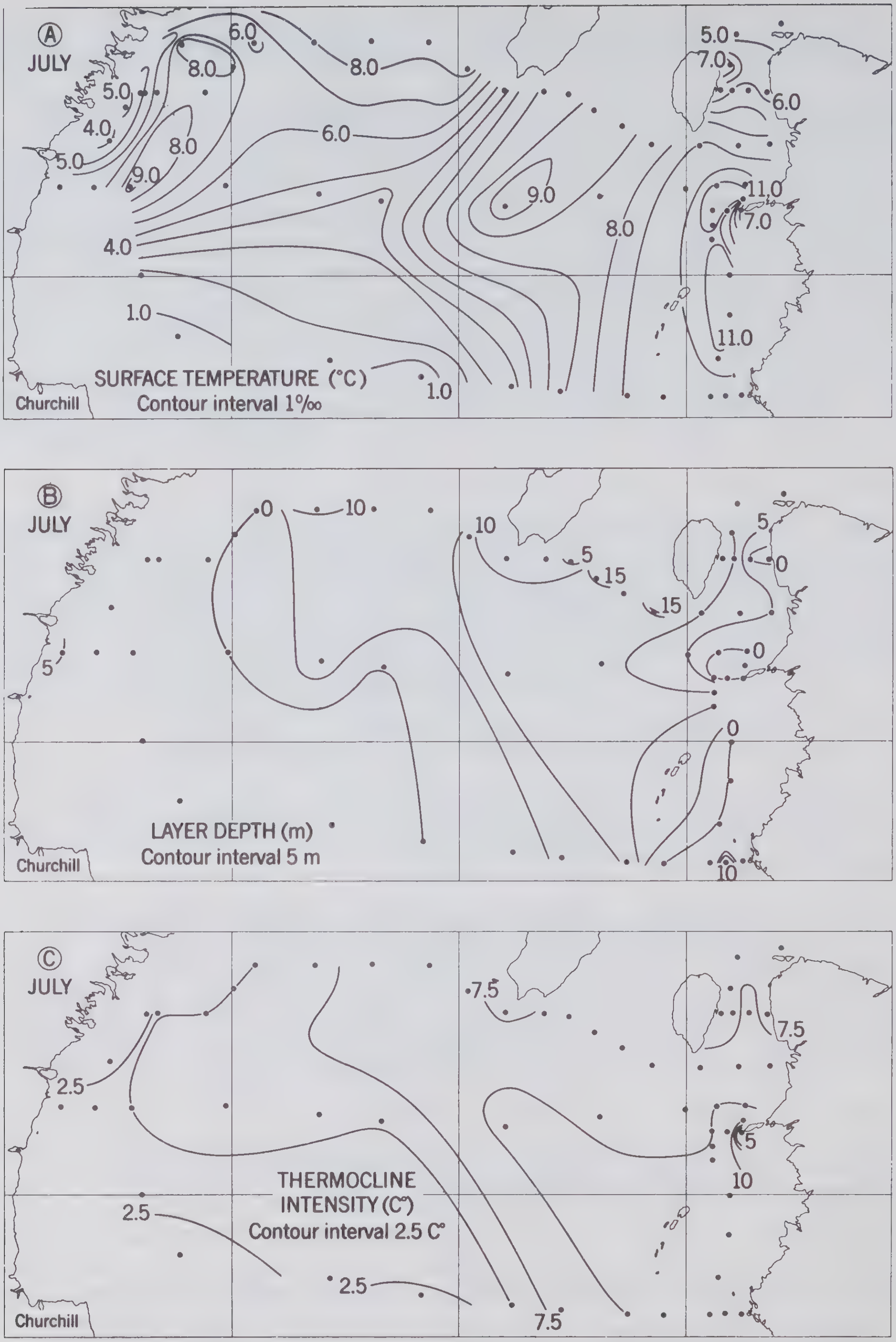


Figure 1. An assessment of the "Theta" bathythermograph observations of July 22 to July 31, 1961 for features of the near-surface temperature structure. (a) Surface temperature (°C). (b) Mixed layer depth in metres. (c) Thermocline intensity as represented by the temperature decrease (C°) in the 25m depth interval below the top of the thermocline.

differences could reflect the influence of a recent or existing ice cover and, while Hachey does not indicate the extent of the ice during the 1930 season, it does appear (Anon., 1931, c) that it may have been a light ice year, at least with respect to the route to Churchill, and as well "Acadia" did not encounter ice from August 2 when she entered Hudson Strait to her arrival at Churchill on August 17.

Data obtained in the latter half of September in 1961 and 1962 (not shown) indicate the existence of a west to east gradient (isohalines north and south) across the central part. It seems that as the open or ice-free season progresses and the initial effect of the addition of fresh water from melt water decreased in relation to that of run-off, the gradient becomes latitudinal. Data in the north observed in September in 1962 (see inset ii) indicate lower values than in August; a similar observation may be made with regard to the 1930 "Loubyrne" data, and the late October observations of "Labrador" (inset iii).

The surface temperature distribution in August (Fig. 2, b) can also be dominated by the existence of an ice cover. During this month in 1961 ice occurred in the southwest and as a consequence temperatures only a few degrees above the freezing point were observed there. Across the central area values were generally greater than 5°C and approached 9.0°C north of Churchill. To the south values were generally less than 5°C except in Richmond Gulf where temperatures warmer than 17°C have been observed; no doubt similar high temperatures occur locally in other sheltered areas. Generally at this time and as described earlier a seasonal mixed layer and associated thermocline occurs in which temperatures in excess of 8°C have been recorded. Locally and particularly near the coast the usual evolution of the seasonal thermocline may be influenced by the addition of fresh water or by mixing due to tidal activity. The former leads to a shallow and stable surface layer so that relatively high temperatures can be attained, as in the northwest area of low surface salinity in July at which time temperatures reached 8°C (see inset i to Fig. 2, b). Tidal mixing breaks down the thermocline and distributes the heat absorbed at the surface over the depth so that surface values may be relatively low. This process is believed to be significant in the extreme northern part of the region where values were generally less than 5°C. In the approach to James Bay surface temperatures can remain relatively high, to 5°C (inset ii), into October, with peak values of at least 9°C (inset iii) occurring in late August.

The distribution of surface temperature observed in "Loubyrne" in 1930 (Hachey, 1931, b) is quite different than that indicated here in Figure 2, b. Temperatures were highest in the southwest and west, about 9°C, but were generally high throughout. As with the distribution of surface salinity, there does not appear to be any effect due to a recent ice cover or accumulation of ice as in the 1961 season.

The salinity distribution at 50m (Fig. 3, a) indicates the existence of higher salinities at this depth than at the surface, although the main features of each are the same. High values, about 32.7°/00, occurred in the north from Nottingham Island to Roes Welcome Sound where values approached 32.9°/00 in a ridge or high extending to the southwest. Low values, about 24.0°/00, occurred in James Bay, and in the region of the Belcher Islands values were generally less than 30°/00. Over most of

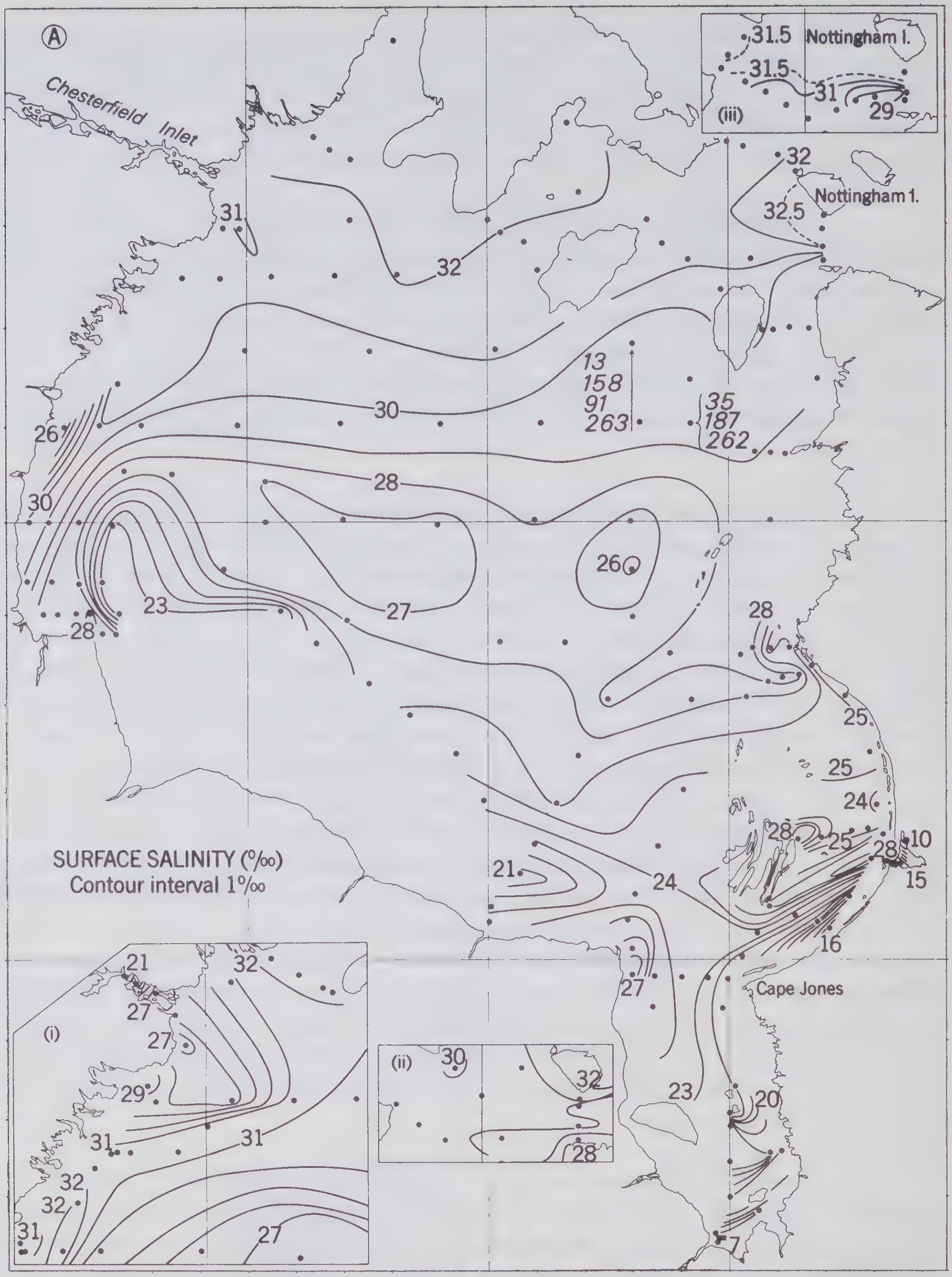
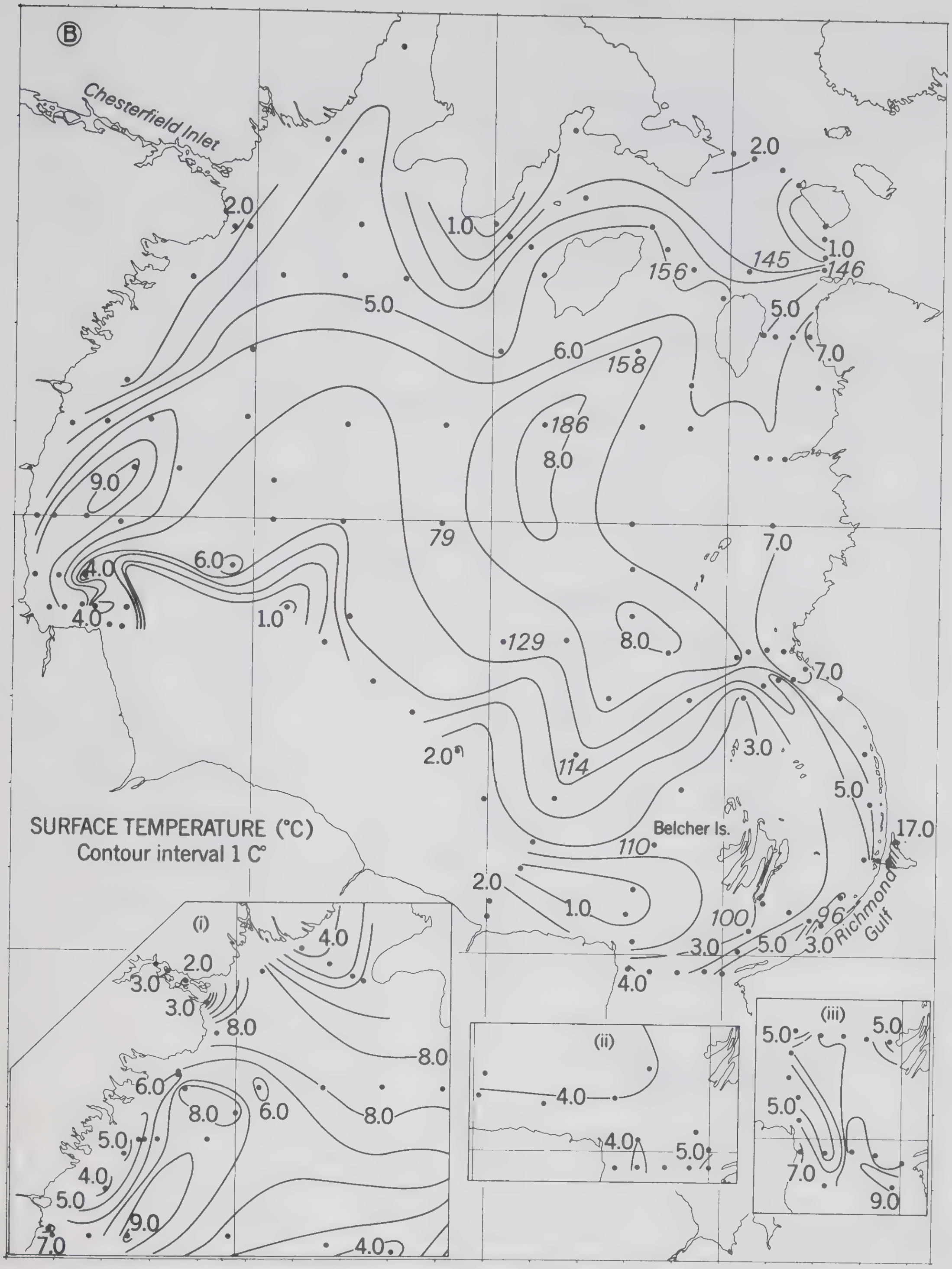


Figure 2. Distribution of salinity and temperature at the surface from data observed during the open season and derived from a number of sources. The closed circles or dots indicate the positions at which stations were occupied, and where observations were generally made at the surface and intermediate depths to the bottom. (a) Surface salinity (°/00). The numbers in italics indicate the positions of the stations referred to in Figures 5 and 9. Inset diagram (i) is from "Calanus" and "Theta" observations early in the 1961



season. Inset (ii) is from 1962 "John A. MacDonald" observations, and inset (iii) is based on the 1955 survey of "Labrador". (b) Surface temperature (°C). The numbers in italics indicate the positions of the stations of Figure 4. Inset diagram (i) is from "Calanus" and "Theta" observations early in the 1961 season. Inset (ii) is based on "Theta" stations of October 1 to 3, 1961, and (iii) on "Calanus" observations from August 26 to September 22, 1959.

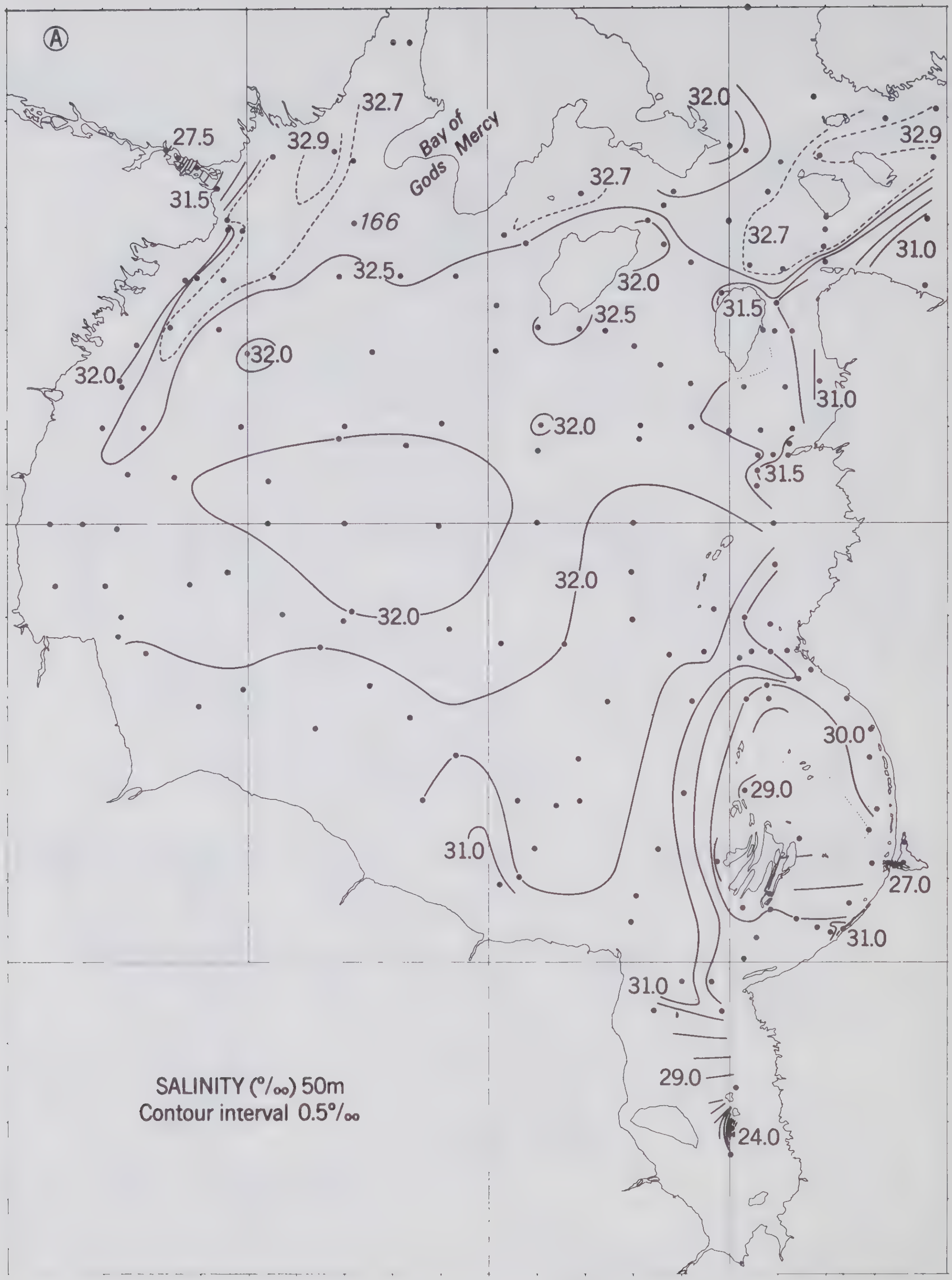
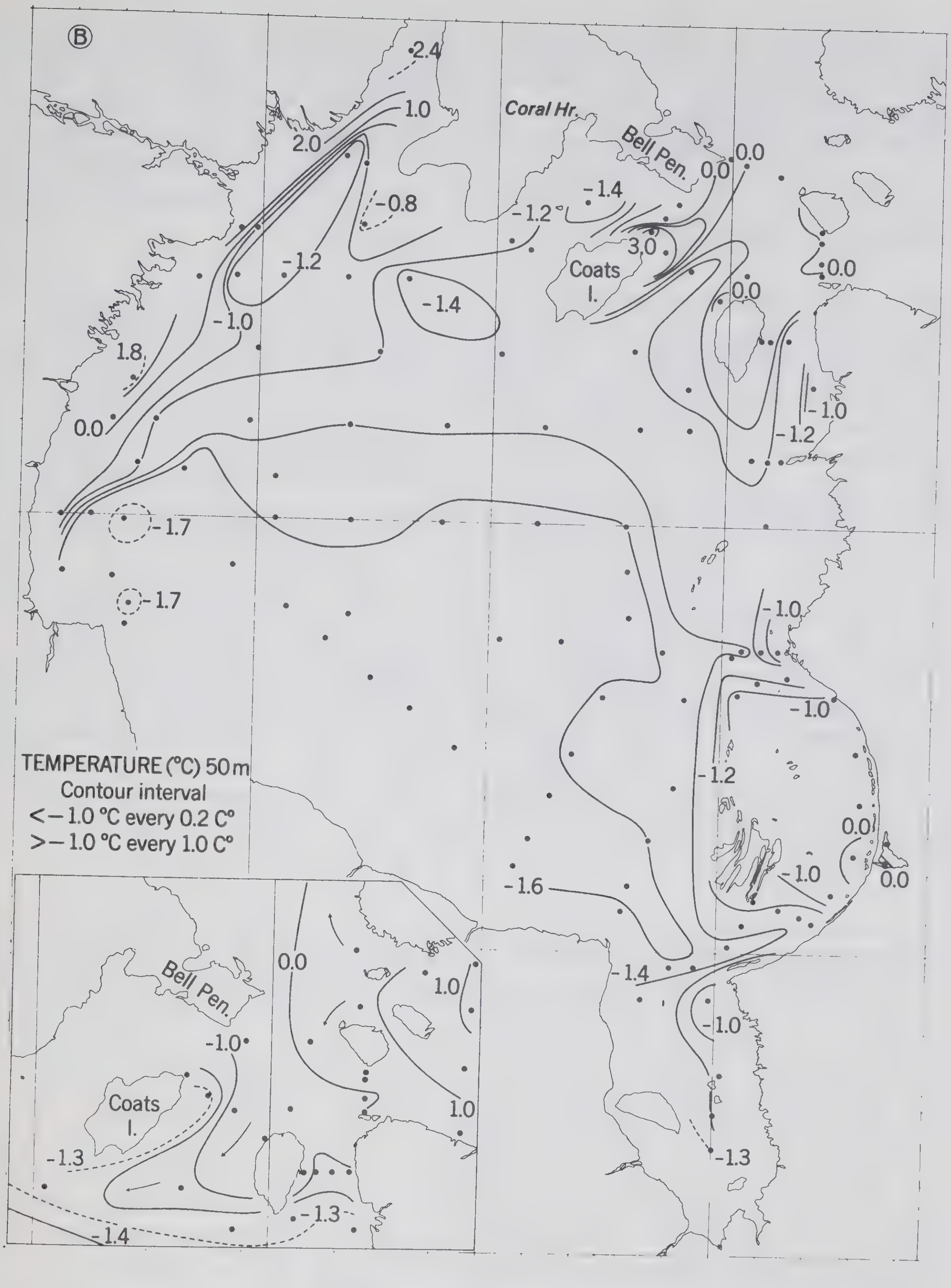


Figure 3. Distribution of salinity and temperature at a depth of 50m. (a) Salinity (°/00). A composite presentation of data from various sources.



(b) Temperature (°C). An interpretation of data obtained during August. The inset is based on data of late September.

the region salinities were in excess of 32°/00, but the central area was marked by the existence of a cell of less than 32°/00. The presentation of Hachey (1931, b, p. 98) does not indicate the existence in 1930 of the ridge of high salinity or the low salinity cell, and again the values in the north appear less than in 1961; otherwise the configurations are quite similar. Data in the northwest were not observed in 1962, but across the central area a strong west to east gradient was observed with values to 33°/00 just north of Churchill to about 31°/00 along the east coast.

At 50m (Fig. 3, b) the temperature was colder than -1. 2°C over most of the region including James Bay. Minimum values of -1. 7°C were observed seaward of Churchill while south of the latitude of Churchill a temperature colder than -1. 6°C was extensive over all the open region, even into the approach to James Bay. Values increased to the north reaching values in excess of 2. 0°C in Roes Welcome Sound and 3. 0°C off northeast Coats Island but with an apparently isolated low value just south of Coral Harbour where a water colder than -1. 4°C was observed. In the northwest the data are limited and somewhat contradictory; in the diagram a region of cold water, to -1. 3°C, is indicated in the region of the high salinity ridge there of Figure 3, a. Otherwise, close to the coasts temperatures were generally warmer than in the central region. Of special interest is the region of relatively warm water off the northeast of Coats Island and another off Bell Peninsula which are indicated in the diagram as being continuous. Whether this reflects the actual situation is not clear, for as Campbell (1959, p. 12) indicated the currents in the region of Evans Strait are complex and likely exhibit considerable variation. As will be suggested, there is evidence that the major movement in this region is into Hudson Bay so that water adjacent to Nottingham Island and Foxe Channel may enter Hudson Bay. In the inset diagram to Figure 3, b, an observed temperature distribution at 50m in summer is presented and in which is indicated that probable persistent direction of water movement at the depth.

Distributions observed in August in a section from southern Hudson Bay (station 96) through central Hudson Bay to the northeast (station 146) are presented in Figure 4. The seasonal thermocline and surface isothermal layer was a pronounced feature over most of the section, but notably not in the northeast where the isotherms were not concentrated in the thermocline structure.. Immediately below the thermocline temperatures were generally less than -1. 2°C and decreased to about -1. 5°C at the bottom so that the temperature structure at depth was relatively isothermal. A slightly colder water occurred at intermediate depth over a portion of the section causing a small minimum in the structure in the area. The surface water was near isohaline over much of the section except in the southeast, while immediately beneath this a marked gradient occurred. The gradient or halocline occurred at the same depth as the thermocline although some salinity increase with depth continued to the bottom. Relatively high values were observed in mid-Hudson Bay (stations 79 and 129) forming a dome in the salinity and density distributions there. The dissolved oxygen content was close to saturation in the surface layer defined by the temperature and salinity. Toward the bottom of the thermocline where it existed, values increased to saturation corresponding to the colder temperatures, then generally decreased to the bottom. Relatively low values were observed below the thermocline at station 158 which, as will be shown in the description of Figure 5, provide a clue to the circulation within Hudson Bay.

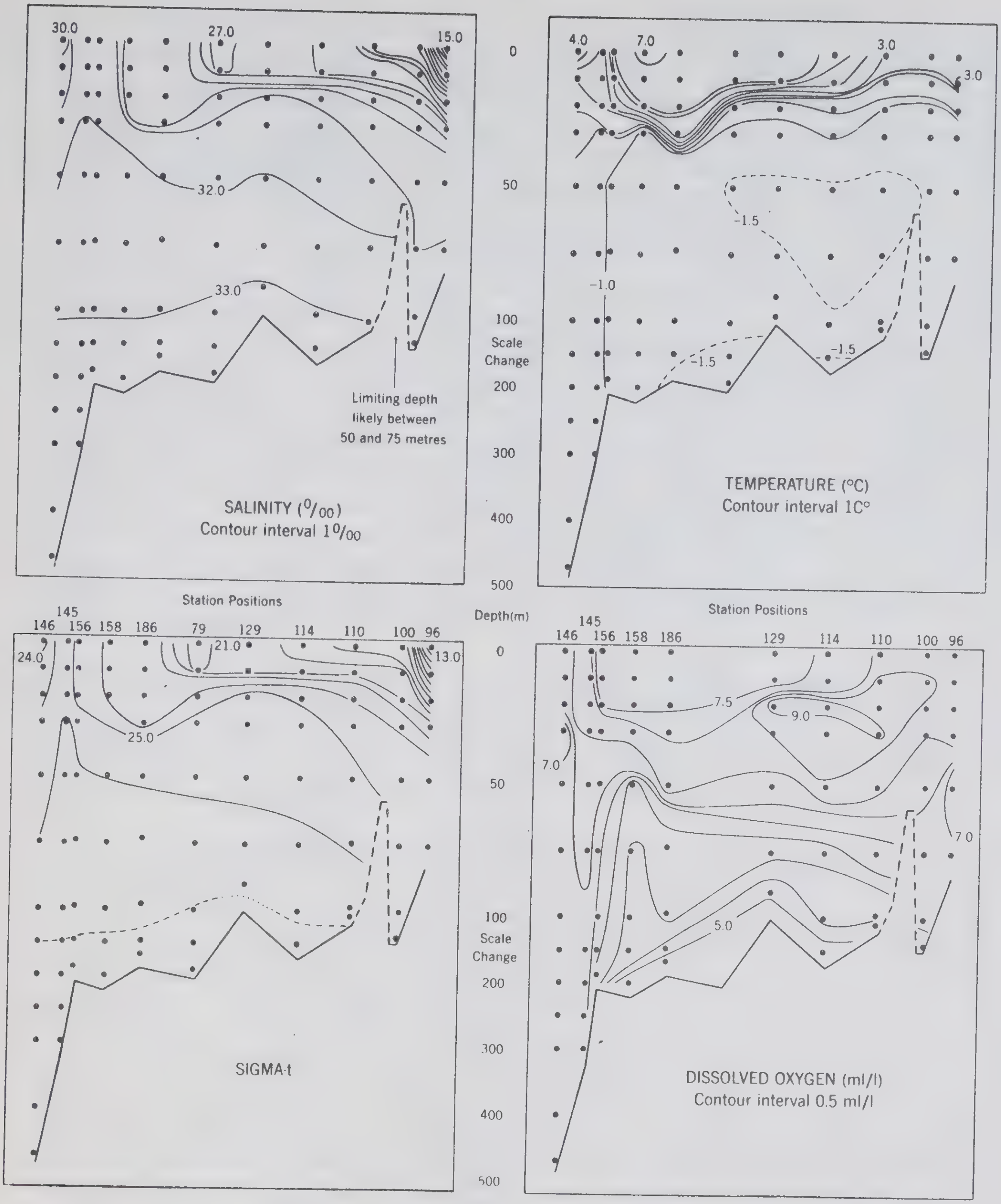


Figure 4. Distribution with depth of the temperature, salinity, dissolved oxygen, and sigma-t in a section from northeast Hudson Bay through the central area and to the approach to James Bay, as derived from data observed in August, 1961. Station numbers are "Theta" 146, 145, 156, 158, 186, 79, 129, 114, 110, 100, and 96; station locations are shown in Figure 2, b. Note the change of vertical scale at 100m and that dissolved oxygen data were not available for station 79.

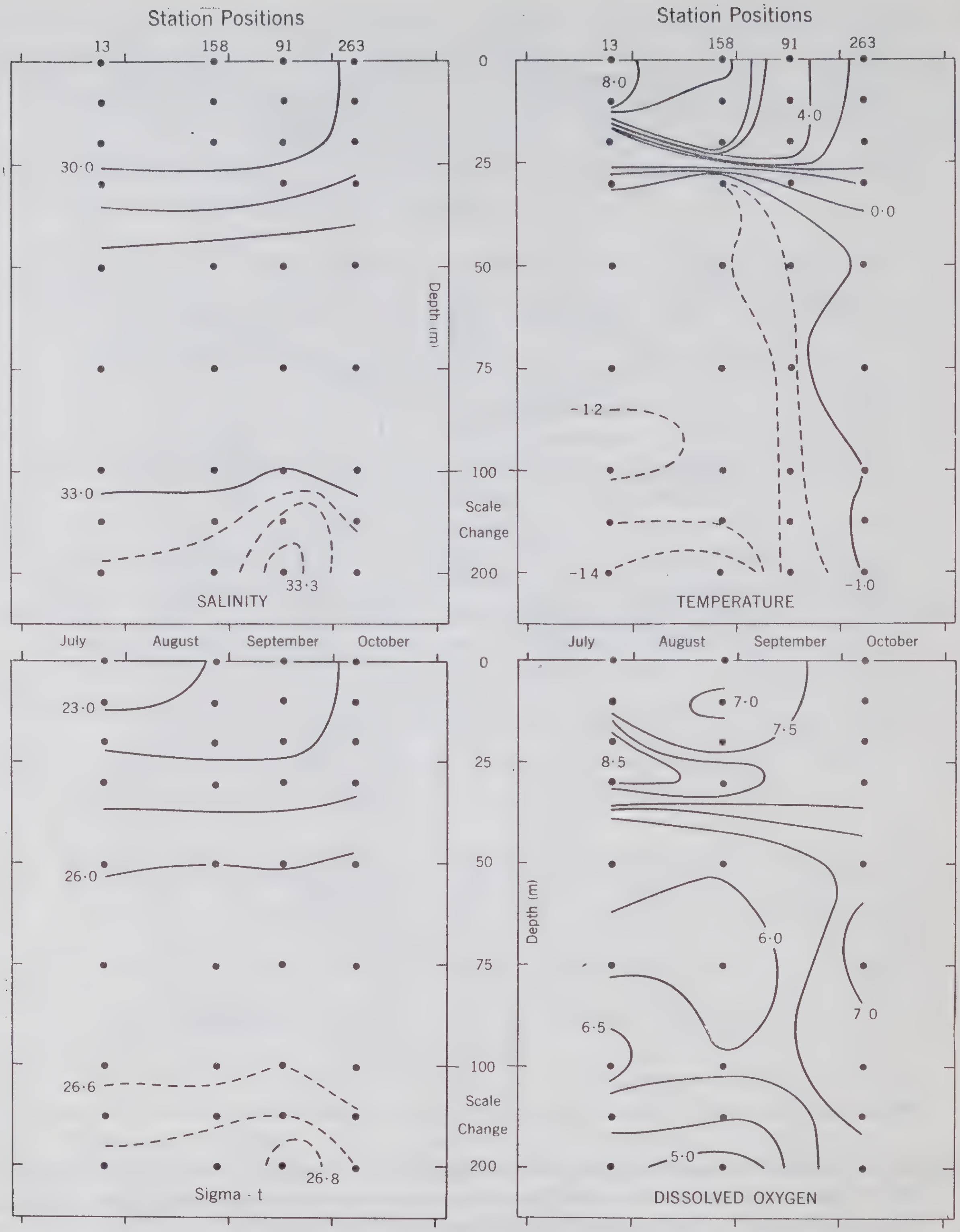


Figure 5. A presentation of data obtained at about one location in Hudson Bay in the seasons of 1961 and 1962. The data was observed at "Theta" stations 13 (61°48' 82°00', July 24), 158 (61°48' 81°58', August 27), 263 (61°50' 81°56', October 7), and "John A. MacDonald" station 91 (61°50' 81°58', September 16, 1962). Note the change of vertical scale at 100m and that dissolved oxygen determinations were not available for station 91. The location of the observations is shown in Figure 2, a.

The values at depth shown at both the north and south ends of the section are determined to an extent by separation due to sills or limiting depths*. The northern part extends into Hudson Strait where the deep water properties (at station 146) certainly suggests the existence of a sill, likely in the section between Nottingham Island and Charles Island. As mentioned in a later section in regard to the waters of Hudson Bay proper, it seems likely that limiting depths do not exist in the approach to Hudson Bay. The situation is not clear for in addition to the general paucity of bathymetric data the usual interpretation of property distributions is complicated by strong tidal mixing. Nevertheless study of the data has led to the conclusion that in the passages between Southampton and Coats Island, Coats and Mansel Islands, and Mansel Island and the Mainland, the limiting depth is not significantly less than that indicated by available bathymetric data. In the south the depth data (Canadian Hydrographic Service chart no. 5003) has been interpreted as indicating a limiting depth of about 100m in the region southwest of the Belcher Islands to Cape Henrietta Maria. Study of the salinity data in the vicinity of stations 96 and 100 in the section where a maximum value at depth of 31.8°/00 was observed, indicates that a limiting depth of between 50 and 75 is more likely. The assumption is made that between the Belcher Islands and the mainland to the east, at least one area of limiting depth exists less than 50m. A likely region for such a depth is that from Tukarak Island eastward, encumbered as it is with islands and shoals.

Extreme values of both temperature and salinity were observed close to the bottom in the northwest. In 1961 these were 33.4°/00 and -1.8°C. Higher salinity within Hudson Bay to about 33.5°/00 has been reported for the "Haida" data of 1948 (Bailey et al., 1951) and for the "John A. MacDonald" data of 1962.

It is proposed now to describe the distribution of properties at one location in the section of Figure 4, i.e., at the location of station 158 where three stations were occupied in 1961 and one in 1962. The data are presented in Figure 5. Prominent in the observed temperature and salinity distributions in time are the halocline and seasonal thermocline. Below 50m the salinity distribution would be quite featureless were it not for the high values of 1962; further suggestion that the deep water salinity was relatively high at that time. The temperature data below 50m are more consistent and, in addition, indicate a change to a warmer water by September with a continuing development into October. The dissolved oxygen data also strongly suggest a change of water as higher values occurred throughout the depth after August. The condition of coldest water with least oxygen appears to have occurred in August, although temperatures in August were only slightly colder than in July. The low oxygen of August suggests a greater influence at the location of a water predominately from inside Hudson Bay. Conversely, the relatively warm water of high oxygen content of October suggests a greater influence of water from outside Hudson Bay, i.e., from Hudson Strait. It does not appear possible to predict with reasonable certainty the progression of events in the deep water at the position after October and to July. This

*The term "limiting depth" is used to indicate situations where depth may limit the connection between bodies of water (Sverdrup et al., 1942, p. 147) in the same sense as does a sill but where a well-defined sill-basin relationship may not exist.

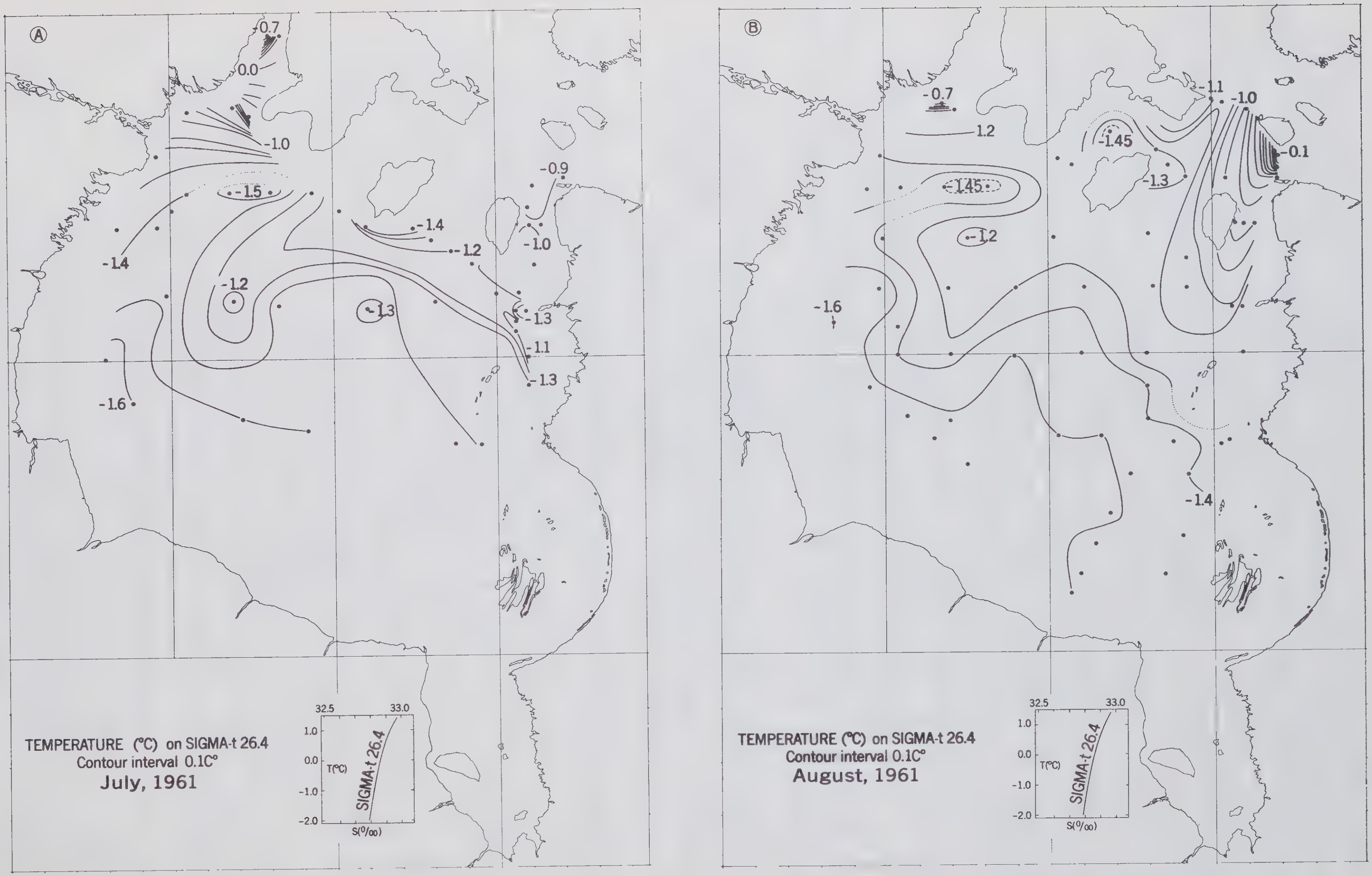
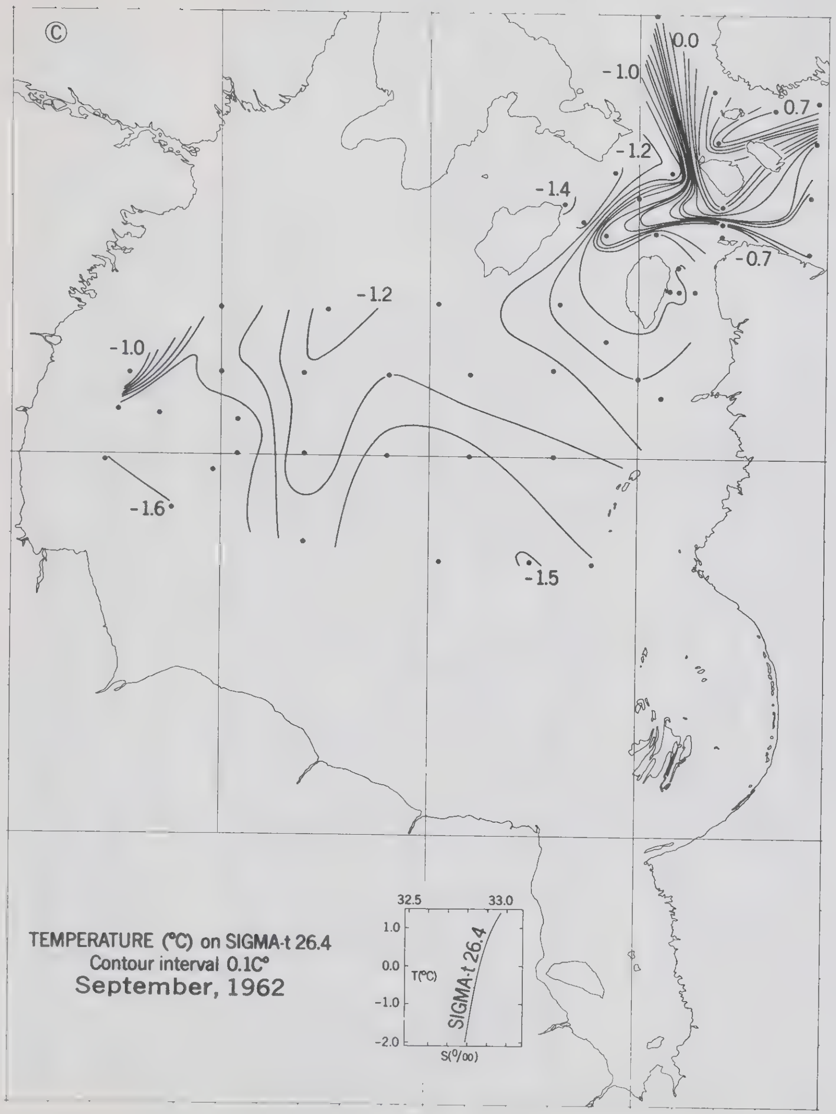
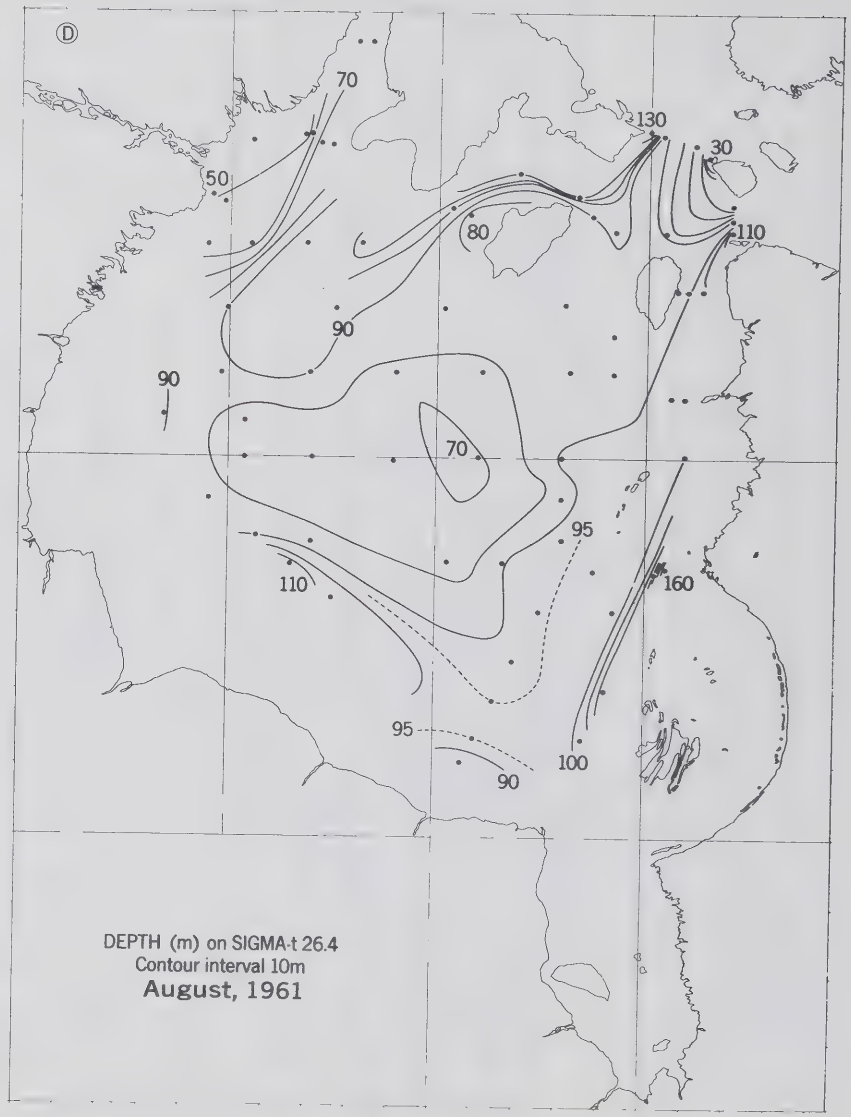


Figure 6. The distribution of temperature (°C) and depth (m) on the sigma-t surface 26.4. (a) Temperature in July, 1961. (b) Temperature in August, 1961.

(c) Temperature in September, 1962. (d) Depth in August, 1961. After Montgomery (1938) it is assumed that the direction of movement is anti-clockwise about a dome or high of sigma-t.





is particularly so because, as mentioned, the origin or source of the dominant water type varies in unknown fashion and as well the extent of such changes are not known for Hudson Bay. However, that marked changes can occur as late in the year as November in Hudson Strait is indicated in 1955 data observed in "Labrador" to which Campbell (1958) drew attention. It seems then that if the influence of Hudson Strait water were to continue beyond October at the location of station 158, a further warming could be expected at least into November and perhaps even later.

At greater distance into Hudson Bay from the position of station 158 the effect in the deep water becomes less discernible, and in the balance between the various influences receives expression in an apparent semi-permanent tongue-like distribution of property on any surface examined. This is indicated here in the distribution of temperature on a constant density surface (Fig. 6, a, b, c). The tongue comprises a relatively warm water extending to the west and south from the vicinity of Coats Island and western Hudson Strait. The September (1962) distribution indicates a warmer water in the tongue than occurred in July and August (1961), which is considered to reflect the same cyclic trend to a warmer condition late in the season described above for that data observed at the location of station 158 (Fig. 5). The 1961 data are sufficiently extensive to indicate (Fig. 6, a, b) that the tongue is bounded in the northwest by a partly isolated cell of colder water. This water appears to be part of a cold, to -1.8°C, and saline, to 33.4°/00, deep water observed in the locality in 1961, although the water in the cell contains less oxygen. The 1962 data do not provide information in the area of the observed cell but do indicate that the cold and saline deep water existed, but perhaps further to the southwest. The low temperature of this water suggests that it is a product of winter influences, probably occurring further to the north within the system. The envisaged mechanism of formation of the water is described later, but it is to be recognized now that such a water may not be widespread* in the north late in the season. Here in September (Fig. 6, c) the temperature on the density surface attained 0.7°C at least and was warmer than -1.0°C to the bottom (not shown). As interpreted in the figure it is processes in this area which contribute most of the warm water to the tongue.

The orientation of the tongue, and of the cold cell, suggests that the predominant water movement into Hudson Bay occurs there in the northwest, and that the subsequent movement is an anticlockwise one, at least in that part unrestricted by limiting depth. The general distribution of dissolved oxygen supports this, for values in the north and west are higher than those in the east and south. This is illustrated in Figure 7 in which is presented the distribution on the surface of constant density as observed in August, 1961. Lowest values, less than 5ml/1, were observed off Smith Island and eastward of the central shoal. It is conceivable that the latter area with depths as shallow as 30m could contribute to the observed depletion through an increased biological utilization associated with the shoal, and through a general restriction of water movement downstream of the shoal. However, the general depletion over the eastern part

---

*The "Labrador" data of 1955 and 1956 indicate the existence of such a water in Foxe Channel (Campbell, 1964), as do the two "Theta" stations 151 and 152 occupied in extreme southern Foxe Channel in 1961 at which time values to -1.68°C were observed.

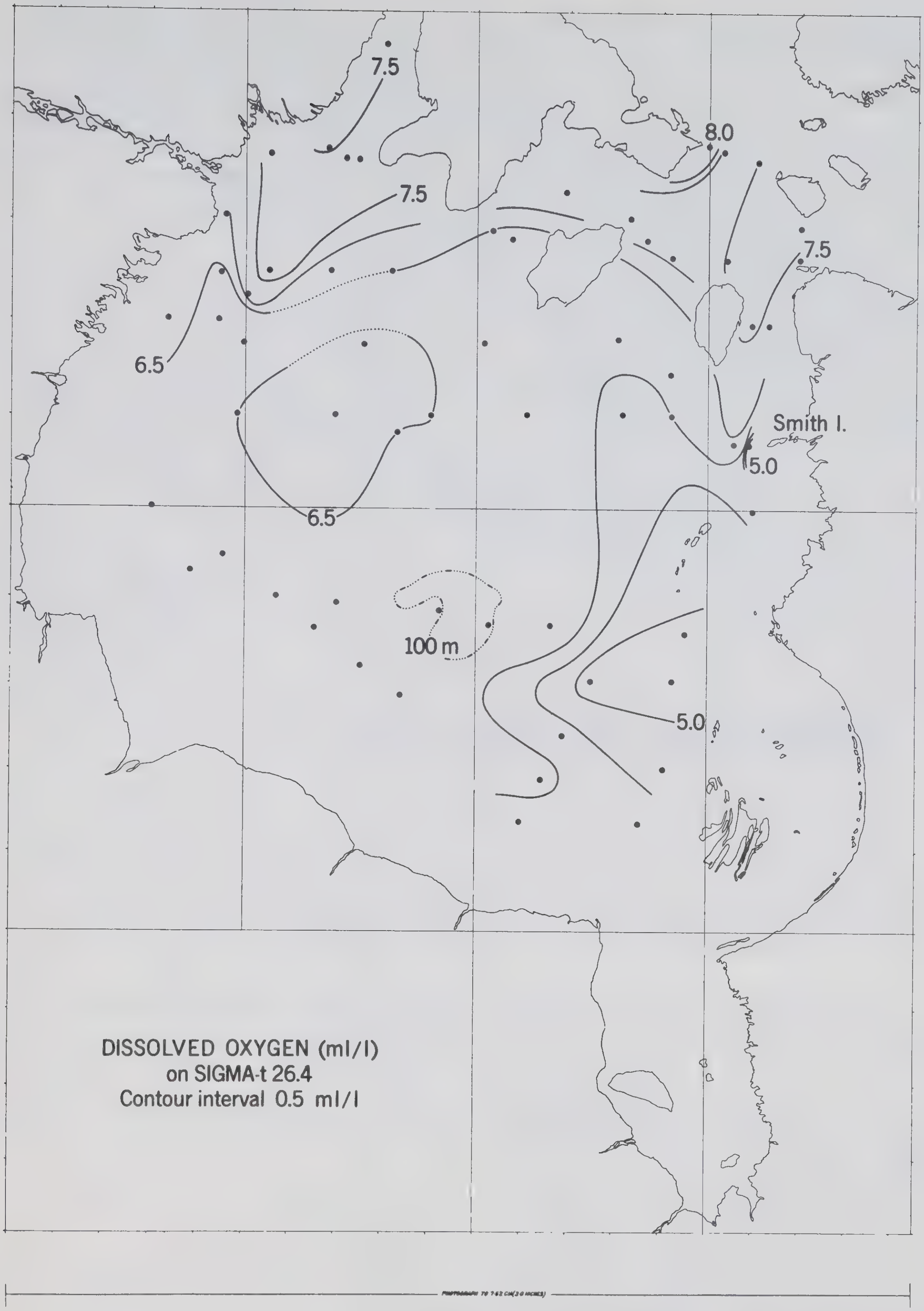


Figure 7. A composite distribution of dissolved oxygen in millilitres per litre on the sigma-t surface 26.4 made up from data observed in the season of 1961 in "Calanus" and "Theta". Indicated is the approximate area of the central shoal of depth less than 100m (from Canadian Hydrographic Service chart 5449).

of Hudson Bay must occur through consumption, and the observation above that the oxygen distribution supports the envisaged water movement invloves an assumption concerning the consumption, as well as the replenishment of dissolved oxygen, over the region. Furthermore the data indicate that throughout Hudson Strait the oxygen values are always high, close to saturation, due apparently to the intense tidal mixing there. Significant depletion would therefore be observed only in Hudson Bay. Such consideration has led to an evaluation of the residence time or age of the water in Hudson Bay (Barber, in review), and to a consideration in later section of the extent to which deeper water is re-circulated within Hudson Bay. In this regard the cell of water to 6.5 ml/1 (Fig. 7) need not have been shown as being isolated, but was as there existed a fairly strong indication in the data that a situation approaching this occurred. It is coincident with a trough in the distribution of depth of the density surface (Fig. 6, d), which in turn suggests a partial recurving to the north of an inward movement already indicated.

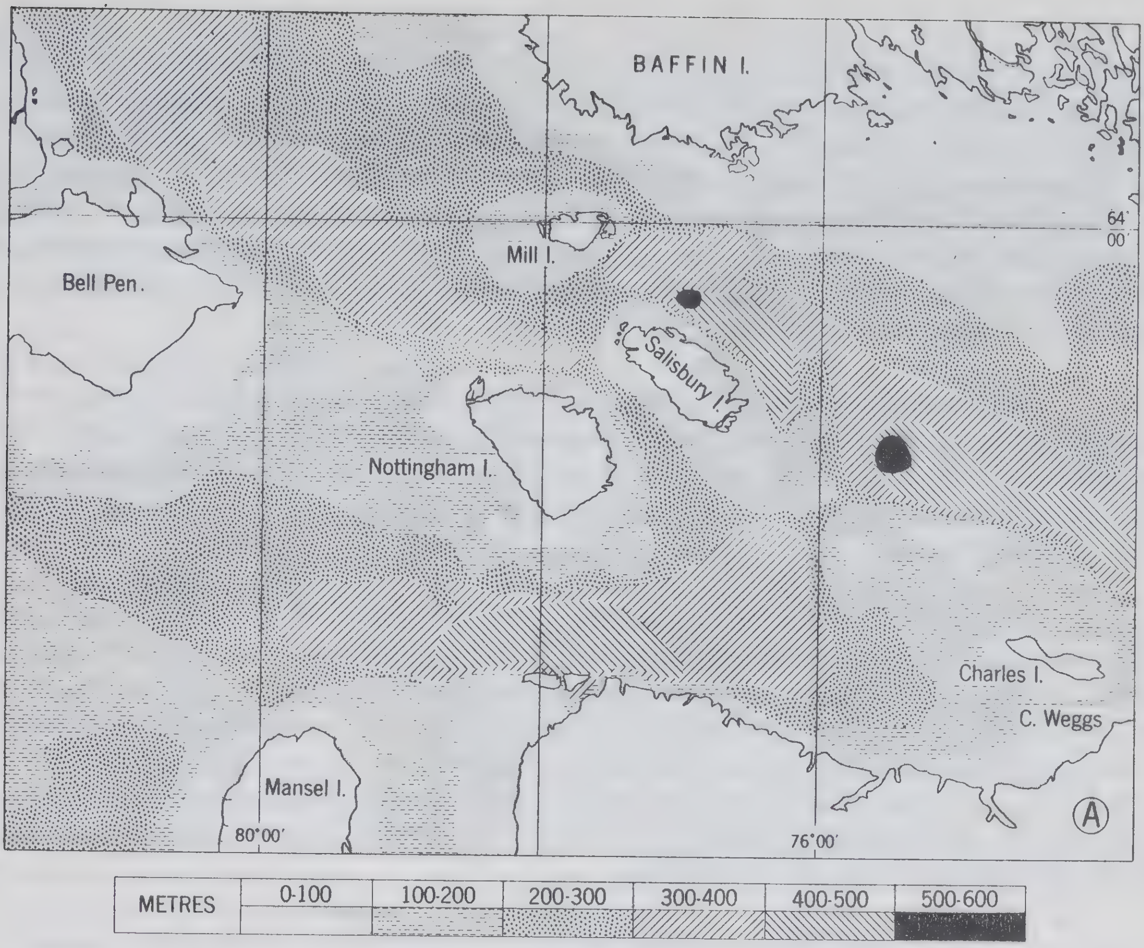
3. FACTORS WHICH DETERMINE THE DISTRIBUTIONS

The distribution of properties within a given body of water is determined by a number of factors; some operate locally, others at a considerable distance. Experience indicates that apart from a surface layer in which seasonal events are observed the property distributions are relatively unchanged with time. This has been shown to be generally true in oceanic areas and comprises one of a number of major assumptions frequently made in oceanic survey and description. In Hudson Bay it is likely that the assumption of a steady state may hold over an extended period, i.e., for a year, although there is evidence that the salinity and temperature of the deep water vary within small but significant limits from year to year. With regard to Hudson Bay the factors include the distribution of depth and current throughout the area and in the connecting channels, the interaction at the surface between the atmosphere and the sea, and the resultant water movement and ice formation, and the influence of fresh water from land drainage and from precipitation. Within the system and in local areas, factors such as tides and tidal streams, and winds can be important.

In the situation relative to Hudson Bay and in the oceanographic context, it is frequently possible to recognize the effect of these influences simultaneously in adjacent areas such as James Bay, Foxe Basin, Fury and Hecla Strait, and Hudson Strait. In consideration of this, occasional reference is made in the following to the "Hudson Bay-Foxe Basin system", or more simply to "the system", to include all of the aforementioned regions.

3.1 Net current and depth

The waters of Hudson Bay with James Bay and Foxe Basin are connected to the waters of the Arctic archipelago and Atlantic Ocean through Fury and Hecla Strait and Hudson Strait respectively. Information concerning the nature of the water movement in Fury and Hecla Strait is fragmentary, but it appears that superimposed on a net eastward transport of about 0.05 to 0.1 x $10^6 m^3/sec$, a variation of speed occurs with tidal period. Collin (1958) reviewed available information and concluded that variations in the net transport would likely occur in response to meteorological effects,



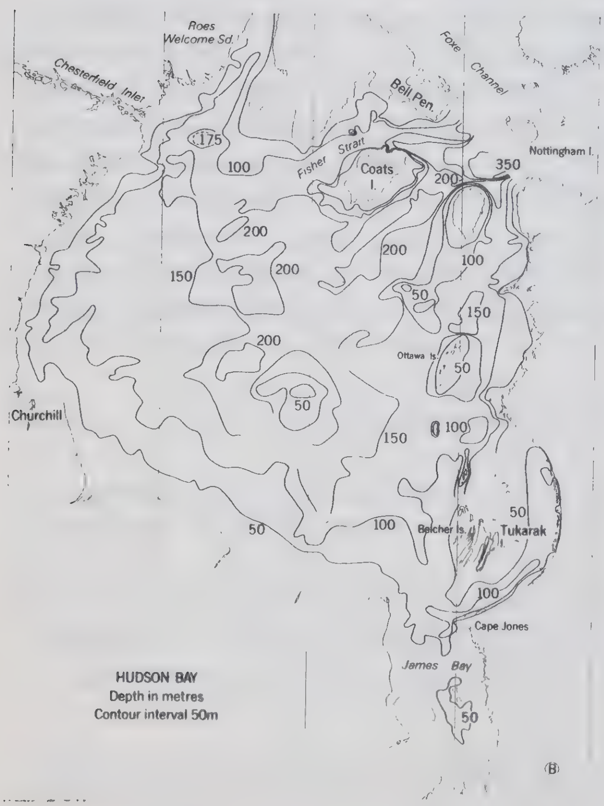


Figure 8.
An interpretation of the distribution of depth in Hudson Bay and approaches derived from information published by the Canadian Hydrographic Service. It may be noted that the contour interval utilized is not the same for each presentation. (a) Northeast Hudson Bay from charts number 5449 and 5450. (b) Hudson Bay and James Bay from charts number 5003 and 5449.

and Barber (1965) suggested that heat gained in Foxe Basin may move to the westward in Fury and Hecla Strait as a secondary effect of the tide. In the longer term it is not clear to what extent the observed net eastward movement into Foxe Basin may be significant. A limiting depth of about 60m appears to exist in Foxe Basin just to the south and east of Fury and Hecla Strait and on this account it is concluded that the contribution to the system must be limited to surface water. It is conceivable* that the volume contribution is significant, certainly to Foxe Basin, but a realistic assessment must await further data and study.

This would include the volume of water and ice and the temperature and salinity of each. The transported water would constitute a meridional movement from north to south so that it likely represents a loss of heat. As well, the net ice movement, if significant, is likely into Foxe Basin, which when it melts would comprise a loss of heat to the system.

The situation with regard to Hudson Strait on the other hand is quite significantly different, for relative to the water of Hudson Bay a limiting depth does not appear to exist within Hudson Strait. The water of Hudson Bay then may be said to be freely connected to the water of the northwest North Atlantic. Depth data are limited however, particularly in the section including Charles, Salisbury, Mill, and Baffin Islands, (Fig. 8, a), where it seems certain that a sill does exist relative to the very deep water south of Nottingham Island. A depth of about 200m appears to exist between Coats and Mansell Islands (Fig. 8, b).

Within Hudson Bay, and particularly in the Belcher Island area, some restriction does occur because of limiting depths with, as has been shown, quite distinct oceanographic consequence. Eastward of the Ottawa Islands a maximum depth of about 140m occurs but the limiting depth in either direction around the islands is not considered to be significant. A large area exists to the westward of the Ottawa Islands where bathymetric data are even fewer than elsewhere in the Bay. Here, and about the area of the central shoal, considerable doubt exists concerning the actual situation. In certain local areas, such as Richardson Gulf, Omarolluk Sound, and Chesterfield Inlet, restriction due partly to limiting depths has a marked effect on property levels there; Grainger (1960) described Richardson Gulf and Johnson (1965) remarked on the situation which exists in Baker Lake at the head of Chesterfield Inlet.

In Roes Welcome Sound the limiting depth is estimated as 53m.

Because of the free connection to the North Atlantic provided by Hudson Strait the currents and water movement in the latter are of considerable significance. A series of direct observations in the 1959 season in a section across the strait from Wales Island to Big Island have been reported (Anon., 1960), and Farquharson et al., (1960) described the observations. In this it was shown that a dominant movement into the system occurred at the surface along the northern side of the strait and at

---

*For example, the rate of $0.1 \times 10^6 m^3/sec$ over a year represents 1/10 of the volume of a surface layer 50m deep over the entire Hudson Bay-Foxe Basin system.

depth, while a movement out of the system occurred along the southern shore at the surface and to relatively shallow depth. The marked currents in and out of the system are likely the result of the input of fresh water within the system, and the consequent mixing of the fresh water with the underlying seawater. Due to the mixing and with certain usual assumptions it is possible to show (see section on fresh water) that the volume of the inflow should be at least 10 times that of the fresh water content of the surface outflow. Due to the rotation of the earth the outflow takes place along the southern boundary of the strait and the inflow along the northern boundary.

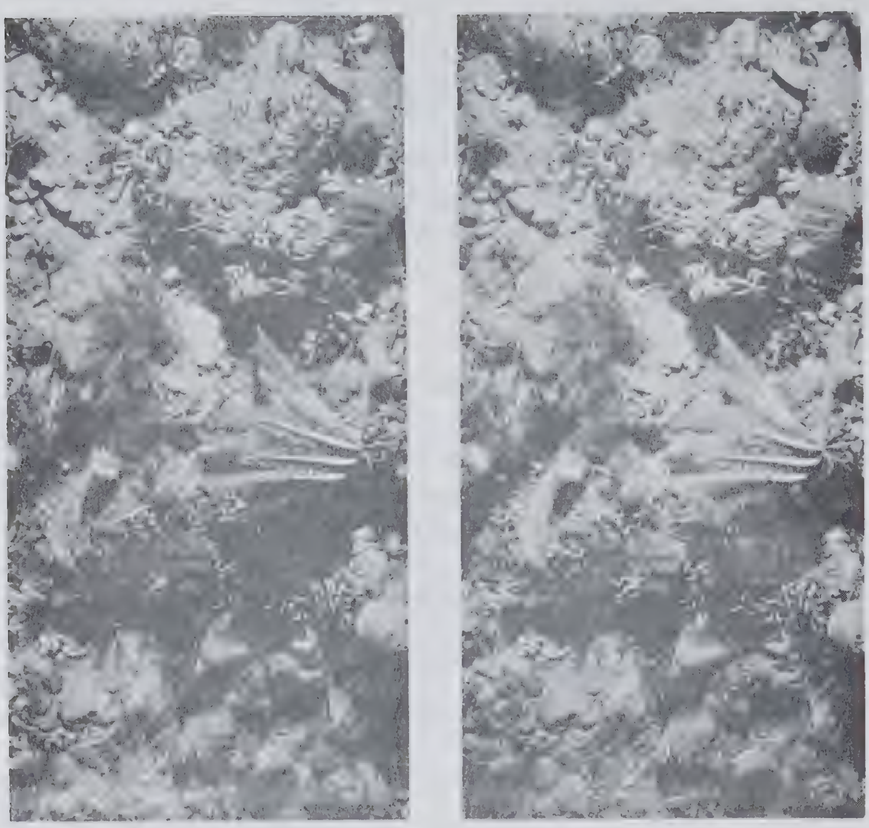
3.2 Tidal streams and tidal mixing

The tidal streams in Hudson Strait and adjacent parts of Hudson Bay have been described as "strong" (Anon., 1954, p. 11), although few direct measurements are available. In the eastern approach to Hudson Strait a strength of 5 knots occurs (Anon., 1954, p. 12) and speeds to 1.5 knots have been reported (Anon., 1960) in the section Big Island to Wales Island. Here five positions were occupied in the 1959 season and at each, observations were made at a number of depths to a maximum of 330m; the amplitude of tidal streams varied from 0.5 to 1.0 knot. Close to Big Island along the north coast (station F) rates in excess of 1 knot were measured at a number of depths to 290m. Other oceanographic data observed at this position have been utilized (Farquharson, et al., 1960) to indicate the strong mixing effect of these streams. In the western approach, in Foxe Channel, intense turbulence due to tidal streams has been reported (Campbell, 1959, p. 14), and in Digges Sound strengths to 2.5 knots are frequent (Anon., 1954, p. 12). In the section north of Digges Sound including Nottingham, Salisbury, and Mill Islands, the tidal streams have been described as "swift" (Anon., 1954, p. 260), and it is remarked that "there are tide-rips and overfalls in the channel separating Salisbury and Nottingham Islands" (Anon., 1954, p. 261). In Frozen Strait strengths to 3 knots occur (Anon., 1959, p. 73), and in Roes Welcome Sound streams are generally strong (Anon., 1959, p. 52) and are thought to continue so southward to the region between the Bay of Gods Mercy and the shore of Hudson Bay to the west. Suspected strong streams in this vicinity led to the occupation there of "Theta" station number 166 at which photographs of the sea floor were obtained. Conspicuous in many of the exposures, as in the four reproduced in the Plate, is the bottom attached animal* which appears to be quite strongly oriented away from the vertical. This is thought to be due to water movement at the depth.

In these localities strong turbulence accompanies the tidal streams and mixing results throughout the depth. Changes of water property occurring at the surface are soon followed by changes in the same direction at depth. Similarly the properties of a water moving into a region either at the surface or at depth become redistributed;

---

*Dr. E. H. Grainger has advised that the animal "appears to be an echinoderm, class Crinoidea, and .... probably Heliometra glacialis (Leach), a widely distributed form in the Arctic, including Hudson Bay." The animal is usually anchored to the bottom but, while the orientation of the animals in one exposure appear to be the same and likely due to current, it is not certain whether they would lean away from or against a current (but see caption to the Plate).



Plate

A reproduction of four of the exposures at "Theta" station 166 (see Fig. 3, a for location) on August 29, 1961 depth 88m using a type of equipment and a technique described b~~e~~y Edgerton (1964). The side-by-side mounts are the result of simultaneous exposure of film in two side-by-side mounted cameras. The object partly visible in the upper right of the lower right exposure is a magnetic compass mounted on a strut below the camera assembly. Early in the sequence the strut touched the sea-floor and as a result a direction could not generally be obtained from the compass. However, later in the sequence some data are possible and from this it has been estimated that the animals described in the text are oriented 130-310° True. The state of the tide at this time at Chesterfield Inlet was about three hours after low water so that it may be anticipated that the observed water movement was one related to the flood at the locality. This deduction coupled with the information in the Pilot (Anon., 1954, p. 271) to the effect that the flood sets into Roes Welcome Sound suggests that the animals were oriented so as to lie with the current rather than into it.

Table II. A listing of the date of drift bottle releases and the quantities released and recovered, from Hachey (1935).

Lot	Date	Released	Recovered	Per cent
1	Aug. 6, 1930	102	none	0.0
2	Aug. 7, 1930	145	17	11.3
3	Aug. 19, 1930	154	5	3.4
4	Sept. 5, 1930	99	4	4.0

the changes may be such that a distinct water is produced. As will be seen, a feature of the water of Hudson Bay is the existence of a shallow layer at the surface in which considerable seasonal heat storage can occur. In the areas of strong tidal mixing the layer does not develop so that the effect of heat exchange (and other exchange processes including oxygen) at the surface is not confined there, and becomes distributed over the depth. In this way temperature changes at depth in these areas occur which can be strongly coupled to and in phase with seasonal temperature change at the surface. An example of this is believed to have been observed in 1961 in southern Roes Welcome Sound where temperatures to -1.1°C at salinities to 33.4°/00 occurred at "Calanus" station 20 and "Theta" stations 167 and 168. The temperature of this water is believed to reflect a recent warming of about 0.7C°, which is considered to result from the absence of an ice cover early in the season, as described in section 4.4, which allows a considerable absorption of heat by the surface water to be subsequently mixed over the total depth. Thus within Hudson Bay there exists a mechanism which can, under certain circumstances, produce changes in a subsurface water.

The significance within the system of such a change as it occurs in Roes Welcome Sound appears small in comparison to that which might be expected in Hudson Strait. Campbell (1958; 1959, p. 8), while he recognized the extent of the seasonal changes there to the extent that "The marked changes which the temperature regimes have undergone by October (as compared to June) are no less remarkable than the changes which have occurred to the salinity field . . . . " did not stress the influence of tidal mixing. This is not unreasonable for currents are strong in the region, as we have seen, and a number of water masses appear to be involved. For example in summer an input of heat can occur in Hudson Strait through local surface heating, through a movement of warm surface water out of Hudson Bay, and through an inflow of relatively warm oceanic water at depth. In those areas of Hudson Strait adjacent to Hudson Bay the temperatures over the depth observed in 1961 and 1962 indicate a seasonal heat storage which could have been brought about solely by local heating at the surface, and which in turn could have been distributed over the depth by the turbulence associated with tidal streams. In these areas, as in Roes Welcome Sound, the effect of tidal mixing can lead to the formation of a local deep water.

Whether the observed warming late in the season throughout Hudson Strait can be brought about in similar fashion, i.e., through local surface heating and tidal mixing, does not appear to have been assessed. It seems that the extent of an ice cover during the period of insolation can influence the total amount of heat absorbed and the final temperature attained. Similarly in winter, the extent of heat loss at the surface may be influenced by ice cover. The subsequent changes of temperature would be communicated to Hudson Bay by means of the inflow described. It is concluded therefore that there exists a strong probability that variations of water temperature in Hudson Bay could occur through influences operating solely within the system.

3.3 Exchange of energy at the surface

Little definitive material is available as regards the influence of climate on the water of Hudson Bay. The fact that a transition occurs each year from a condition of near complete ice cover to one of virtually no ice cover may be taken as a first

order measure of the influence. It cannot be assumed however that the influence extends directly over the total depth of water for, as in the ocean, density stratification exists which largely prevents this, except in well-defined areas. It has been shown that a stratification due to fresh water from run-off exists over most of Hudson Bay. Within this general feature are observed changes due to the growth and decay of the ice cover and the seasonal variation of heat input.

An attempt to evaluate on a monthly basis the character of the heat exchange at the surface is made in a later section. In this heat budget it may be seen that the values of the exchange terms except for the effective back radiation, vary considerably over the year. The general direction of the change is toward smaller values during the period of ice cover so that the annual net gain and net loss values are smaller than if there were no ice cover. The influence of the ice cover occurs through its relatively high reflectivity to short wave radiation (albedo) as compared to water, and very much smaller effective conductivity as compared to water. The first is effective in reducing the amount of absorbed radiation, particularly in the spring and early summer, while the second leads to temperatures at an ice surface in winter close to air temperature so that the flux terms become much less, and indeed almost negligible. The consequent reduction in the size of the net gain and net loss values brought about by the existence of an ice cover tends to restrict the size of the deficit or surplus which might occur. The small net indicated in the budget then was to be expected, as was the result that it was a deficit. Both are consistent with the conclusion of Hare et al.(1949) that the effect on climate of the water of Hudson Bay is markedly reduced, beginning in the autumn, through the formation of an ice cover.

An asymmetrical ice movement leading to an accumulation within the system as described in section 4.4 could contrive to increase the net gain elsewhere in the system early in the season which generally, but not necessarily, would be followed by an appropriately higher net loss in the autumn. It is the net heat gain at the surface, i.e., the net of the radiative and flux terms of the budget, which is utilized to melt an ice cover and to raise the temperature of the water above the freezing point. As shall be shown it is characteristic that the increase of temperature, and hence the seasonal heat storage, takes place in a mixed layer at the surface.

3.4 Ice cover and the surface mixed layer

Lamont (1949) and Hare et al.(1949) showed that by late winter Hudson Bay can be expected to be very nearly completely ice covered and suggested the likely consequence on the climate of adjacent land areas. Dunbar (1954) provided an assessment of what she termed good and bad years and indicated that by late summer the region is generally free of ice. Mackay (1952) suggested that late spring temperatures were of special significance with regard to time of break-up, Markham (1962) described general features of the pattern of break-up, and Mackay et al.(1965) have provided a detailed examination of long-term data relative to dates of freeze-up and break-up on the Churchill and Hayes Rivers. A significant aspect of recent work relates to that of the Meteorological Branch of the Department of Transport whereby ice conditions are assessed frequently through air reconnaissance as described by Archibald et al. (1962). In the discussion the latter work is utilized in a description of an observed influence

of the ice cover in Hudson Bay during the 1961 season. Major features of the existence of an ice cover relate to the influence on radiative and flux processes and to the redistribution of a portion of the fresh water (and salt) in the sea. These are discussed in this chapter in the appropriate sections. Another consideration of the existence of an ice cover is that it hinders normal sea navigation and as a result winter oceanographic data are few. Nevertheless, it has been possible to interpret available information (see section 4.3) so as to allow a determination of the probable winter surface salinity under the ice cover. The interpretation suggests the existence of a cycle of salinity of yearly period with a maximum in late winter. Utilizing this conclusion and data obtained at a location within Hudson Bay in 1961 a gross estimate of the change in near surface salinity structure has been derived (Fig. 9, a). It is seen that except for the immediate surface value in July, salinity values did not change markedly during the period of observations, and that the main feature of the presentation is based on the inferred winter surface slainity which is estimated to be close to 32°/00 in the locality. An important feature is that as the open season progressed the surface layer showed decreasing structure so that by October a mixed layer occurred to about 40m depth, at which time the salinity at this depth in the layer is close to a minimum.

The increase of surface temperature within a surface layer reflects the direct absorption of radiation by the water and downward mixing of this energy. Generally this energy input reaches a peak about mid-summer declining thereafter to about September at which time the energy input equals the energy output. Subsequently the output increases and input becomes zero to the time that the cooling is such that an ice cover forms. This, as well as a snow cover which may exist, reduces the loss of heat considerably. At this time, and again using the conclusion concerning the surface salinity in winter, it is possible to derive (see section 4.3) the distribution of surface temperature in winter under the ice cover. This, combined with the temperature data at a location in 1961 allows the delineation of the gross temperature structure (Fig. 9, b). A significant feature of the vertical temperature distribution in the ice-free period is the intense gradient or thermocline which occurs below the relatively isothermal surface mixed layer. The thermocline is initiated after the removal of ice and is well-defined and shallow, between 10 and 20m, by August, gradually deepening and disappearing with the onset of renewed ice cover. The maximum depth of the mixed layer occurs later during this period but it does not appear to be possible to define the precise depth. In section 4.3 it is suggested that by the end of the period of ice formation the depth of the mixed layer is between 50 and 75m, so that an increase in depth up to 25m appears possible. The feature could be of particular significance relative to the heat flux under an ice cover, and is mentioned again in the discussion of the heat budget.

3.5 Fresh water from run-off

In the surface considerable dilution occurs due to the addition of fresh water from melting ice and from land drainage. The fresh water being less dense than the saline it forms and remains in a surface layer, so that a marked vertical gradient of salinity, the halocline, occurs between the layer and the deeper water. In some circumstances, and frequently because of the influence of melting ice, the halocline structure may extend to the surface. It appears that the least input of fresh water

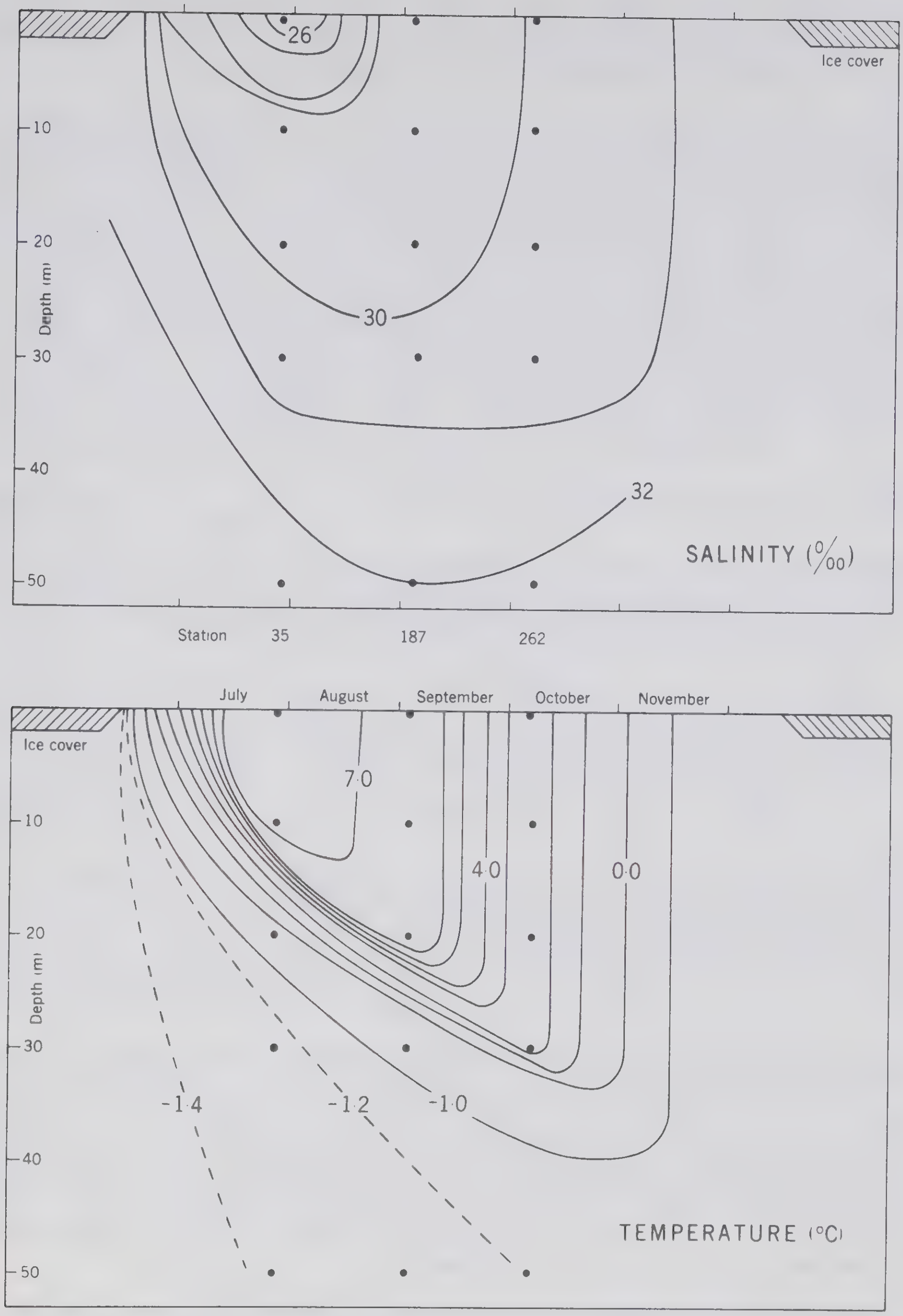


Figure 9. The annual sequence in the near-surface of the salinity and temperature structure as derived from the probable late winter surface salinity and from data observed in 1961 at "Theta" stations 35, 187, and 262 on July 28, September 3, and October 7 and assuming the duration of the ice cover. The location of the stations is indicated in Figure 2, a. The closed circles or dots indicate the depths at each station at which discrete temperature and salinity observations were made. (a) Salinity in parts per thousand (°/00) of salt per thousand parts of sea water. (b) Temperature in degrees Celsius (°C) from serial and bathythermograph observations.

occurs in late winter and the greatest in summer, so that a variation of salinity occurs in the surface layer in association with these influences, that is in association with the growth and decay of ice cover and with land drainage.

This and other aspects are assessed in the crude budget of section 4.1 where it is estimated that it would require three to four years to establish a steady state at the calculated content and annual inflow. On this account it is reasoned that, away from coastal influences, the surface layer would likely not reflect seasonal effects due to such variation of fresh water inflow, thus the observed changes in the layer are probably due to ice. This may be examined in the situation that 150 cm of ice form at an initial surface salinity of 31°/00 and assuming an average ice content of 5°/00, i.e., 5 grams of salt per thousand grams of ice. Considering a cm^2 of sea surface about 150 grams of ice form out of which 4 grams of salt are added to the underlying seawater. This is distributed over an estimated surface layer to about 50m depth so that the salinity there is increased by 1°/00 by about later winter. Subsequently the ice melts causing an addition of near fresh water at the surface and a consequent marked reduction of salinity there. It is concluded that while the fresh water from run-off determines the existence of a low salinity layer at the surface, the growth and decay of ice determine the structure of salinity within the layer.

The characteristics of regions with large input of fresh water have been well documented and are of interest here as it is likely that the circulation pattern which couples Hudson Bay to the Atlantic Ocean is dominated by this through the so-called "estuarine" circulation. In this, fresh water entering at the surface remains in the surface layer and becomes progressively mixed with salt water as it moves seaward. The salt water is provided to the surface layer through upward transfer or entrainment from depths below the halocline i.e., at depths below the surface layer. Here the salt water is maintained through an inflow of water at oceanic salinity. The estuarine circulation then is characterized by a net upward movement balanced by an outflow at the surface and an inflow at depth. It is possible from knowledge of the run-off and the salinity of inflowing and outflowing waters to establish the volume of the inflow. Pickard (1964) reviewed this simple but useful procedure as derived from principles of conservation of volume and salt. With a choice of 30°/00 as the salinity of the outflowing mixed fresh and salt water and 33°/00 as the salinity of the inflowing ocean water, the volume of the inflow is determined to be 10 times the run-off, and the net upward movement of entrainment, assuming the inflow to be distributed equally over the area of the system, is 0.4×10^{-4}cm/sec.

Two independent estimates of the run-off are made in this work. The first stems from the consideration of the amount of precipitation and evaporation over the region as outlined in the fresh water budget which provided an annual value. The other was obtained from the direct current observations in Hudson Strait and in Fury and Hecla Strait, and allowed an instantaneous measure in terms of net transport. The values are in general agreement for as mentioned, the net transport over the year is sufficient to remove the calculated amount of fresh water from run-off which occurs annually.

It has not been possible to obtain realistic values of the run-off of fresh water at time intervals smaller than a year. A maximum probably occurs with the spring and summer freshet and a minimum at some time during the later winter. In certain areas and particularly around James Bay and southeastern Hudson Bay a secondary peak occurs in the autumn due to direct run-off of precipitation falling as rain. The extent to which the variation may be reflected in the distributions of the amount of fresh water is not clear. As is remarked in the fresh water budget the observation concerning the ratio of the annual run-off to storage would suggest that over most of Hudson Bay the seasonal variation of run-off would have little effect. This may be said to be generally true as the limited data do not suggest otherwise. In the northeast, off Digges Island, an increasing amount of fresh water occurred during the period July to October. This does not appear to reflect any particular pattern of accumulation or movement within or out of Hudson Bay, and may be related more closely to events in Hudson Strait.

3.6 Surface wind

> "During the greater part of the year the mean pressure distribution over the Canadian Arctic depends mainly on the extent of the semi-permanent Icelandic low pressure centre and the relatively high pressure ridge extending from the Mackenzie River Valley to the Pole. In winter a trough from the Icelandic low extends west to Greenland. Between this trough and the ridge further west there is a strong northerly circulation." Extracted from Pilot of Arctic Canada (Anon., 1959).

This strong northerly circulation in winter is evident over much of northeastern Canada including Hudson Bay. In the Pilot of Labrador and Hudson Bay (Anon., 1965,b") of the Canadian Hydrographic Service the following remarks are made in the section on the climate of Hudson Bay.

> "The winds are strong in Hudson Bay area during all but the summer months. Hourly speeds of thirty to forty miles per hour can occur in almost any month but the greatest observed hourly speeds, which exceed 60 miles per hour at some coastal stations and have reached 80 in at least one case, have occurred mainly in the autumn months of September, October and November.
>
> "In spring and fall the region is frequently visited by low pressure areas and in consequence the wind direction is variable, although there is a tendency for the northwest quadrant to be favoured. In winter the western edge of the Bay is effected by the continental high pressure area and a very large proportion of the winds are from the northwest quadrant. However, this effect does not extend to the eastern side of the Bay where no one direction predominates. In summer the winds are generally lighter and are variable in direction with a higher proportion of on-shore winds at coastal stations, the effect of local sea-breeze circulation."

It appears then that during the period Hudson Bay is free of ice the winds are light and variable so that marked effects related to wind are not to be expected. There is evidence however that the influence of the high pressure area to the west of Hudson Bay persists into the summer so that the surface wind frequently exhibits a strong northwest component, particularly in the northwest portion. This has been examined for the 1961 season utilizing meteorological observations (Anon., undated a) at six stations around the coast and it is of interest that the frequency of summer winds of direction N, NW, and W at Chesterfield were generally about 50%. A result of such persistent wind would be to produce a movement of the surface water (and ice) which in the NWsector would be dominately out of the region. The required replacement of this water could occur in part through shoreward inflow of deeper water in a condition akin to upwelling. The 1961 data, including those for ice movement, heat storage, and distribution of water property in the NW region, suggest an influence of surface wind occurred as described, but it does not appear possible to assess the significance of the wind relative to other processes which could produce a similar result.

In the short-term and in local areas the surface wind can have marked effect. For example in October, 1961 it was possible to make observations in the vicinity of Cape Smith ("Theta" stations 259-261) immediately after a storm of about two days duration had occurred during which a southerly wind attained near gale force. Seaward of the cape at station 259 an accumulation of relatively warm low salinity surface water was observed to a depth of 80m. The accumulation may be attributed to a secondary effect of the wind (Sverdrup et al., 1942) which in this case would produce a northerly flowing current associated with which would be on onshore movement at the surface.

In winter the effect of the wind at the water surface would be much reduced by the existence of the almost complete ice cover. On the other hand movement of the ice cover could be anticipated over Hudson Bay which would represent the total effect of the wind over some considerable time. Markham (1962) remarked on this possibility but no further information appears to exist.

4. DISCUSSION

4.1 A fresh water budget

Using a technique based on that of (Tully, 1958, p. 531) and a base salinity of 33°/00, estimates of the amount of fresh water in Hudson Bay were obtained from oceanographic data. The formula

$$C = 1 - \frac{\int_{o}^{L} S dz}{S^{*} L}$$

where

C = the fraction of fresh water to the depth L,
L = the depth at which the salinity attained S^*,
Z = the depth in metres, and
S^* = the base salinity = 33°/00 (see note*),

was applied to the observed distribution of salinity at each station and the quantity "CL" obtained. This may be considered as the depth of fresh water that would overlie the salt water. Values of "CL" were obtained for much of the data and the distributions for 1961 and 1962 are shown in Figure 10. The general features of the distribution of fresh water in all the material are,

(1) a relatively large amount of fresh water between the Belcher Islands and the eastern shore, and particularly in the approach to James Bay,

(2) a possible increase in amount of fresh water in the approach to James Bay in October over that in August,

(3) relatively little fresh water along the northwest and north coasts except close to Digges Island, and

(4) increasing fresh water as the open season progressed in the approach to Hudson Strait.

During the month of August the total volume of fresh water was calculated to be about $3.57 \times 10^{12} m^3$. This is equivalent to an average fresh water depth of 4.8m over Hudson Bay. Fresh water occurs in Hudson Bay because of direct precipitation, condensation at the surface, ice melt, advection with salt water and as ice from adjacent areas, and run-off; it is removed through evaporation, advection, and ice formation. Other effects such as ground seepage and diversion by man are assumed negligible. It is also assumed that precipitation over Hudson Bay is the same as that over the surrounding land (which may be more nearly correct in winter than in summer). Charts of the monthly precipitation for Canada in 1961 (Anon., undated) have allowed assessment of the total volume of precipitation falling in the Hudson Bay drainage area as $1.66 \times 10^{12} m^3$ in that year. It was not possible to examine** in any but gross way the extent of evaporation and transpiration over the area, but it seems reasonable

---

The base salinity has not been derived from detailed analysis of the salinity distributions and may eventually prove an inappropriate choice. At those stations (positions) where the salinity was less than S at the deepest depth of observation, "L" was taken to be this depth.

**Directly applicable data does not appear to be available for this determination. The estimate used above was obtained from the study of Cavadias (1961) and applied to the monthly precipitation data for 1961. It seems probable that the effect of evaporation and transpiration is larger than estimated here so that the annual run-off would tend to be less than that estimated, and the accumulation interval for the fresh water in Hudson Bay would be longer.

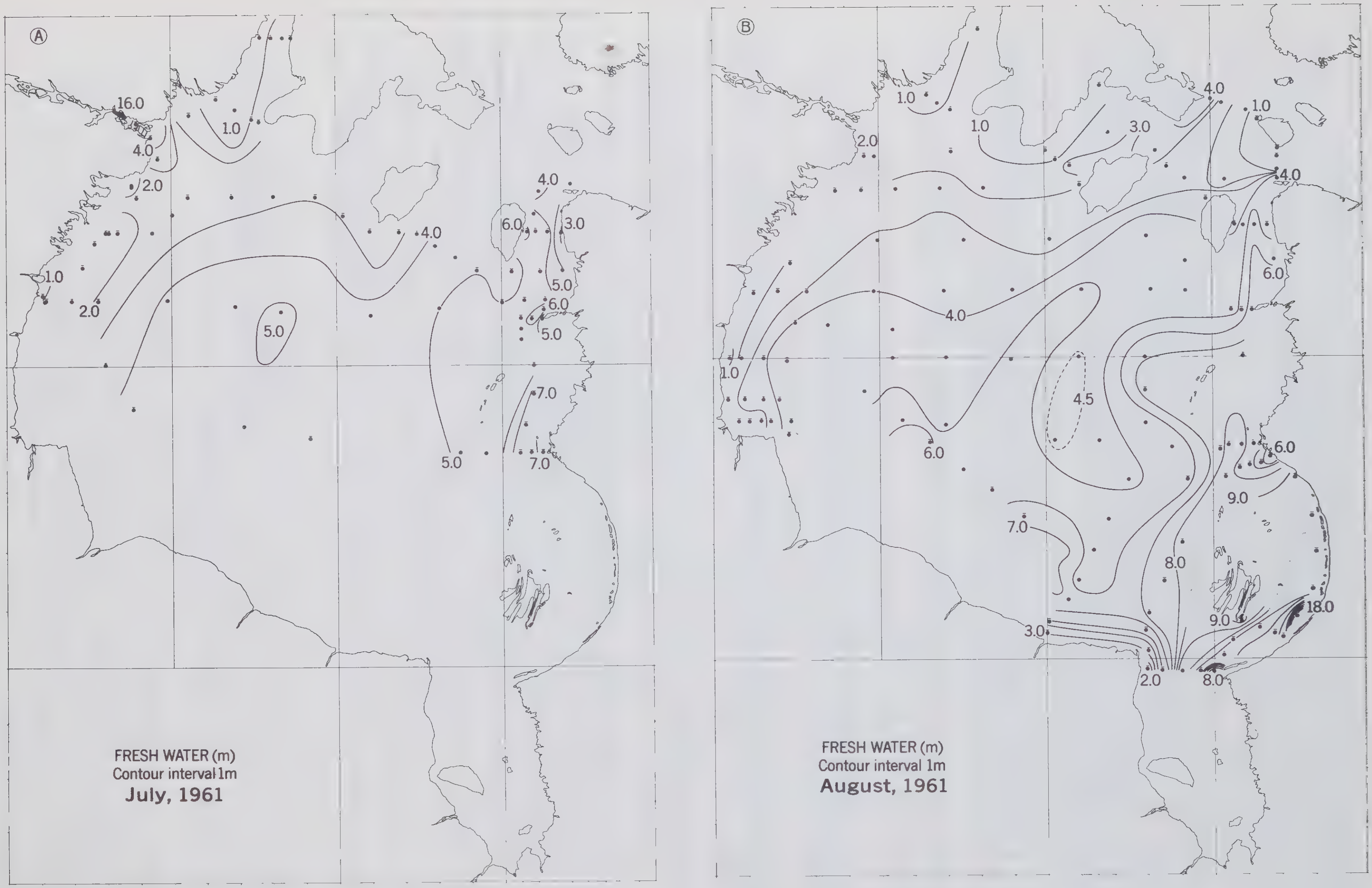
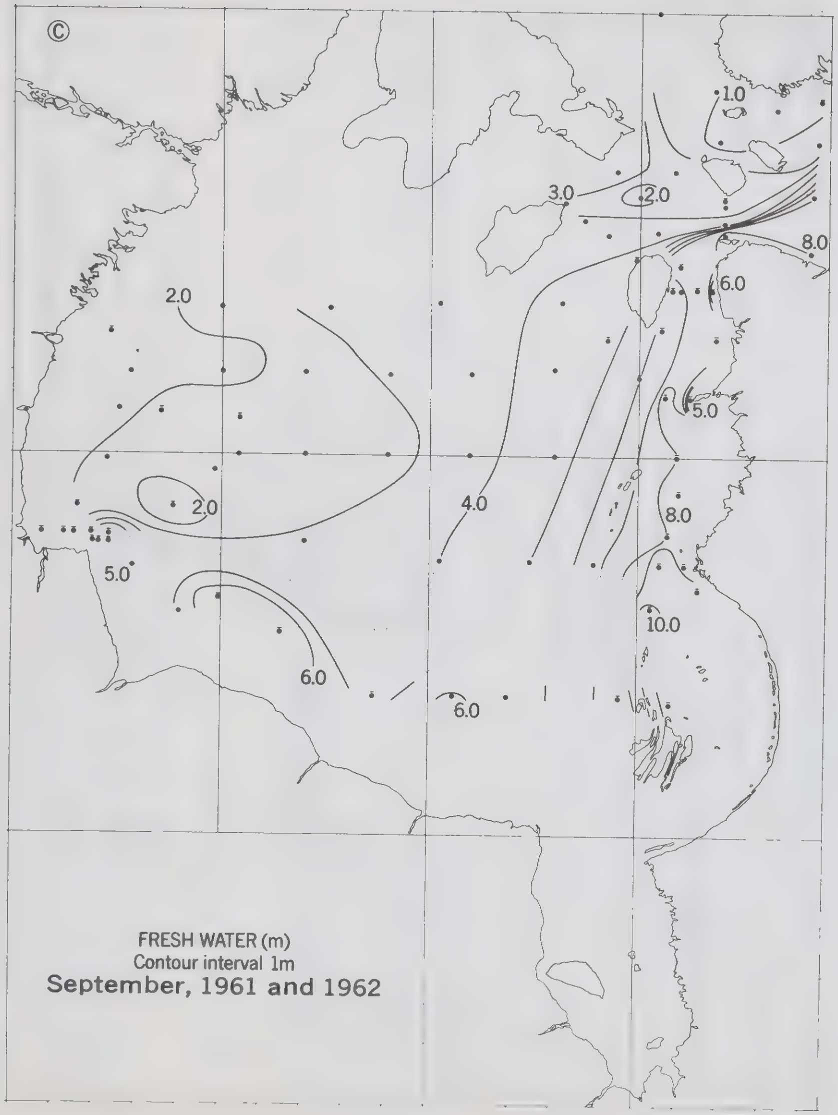
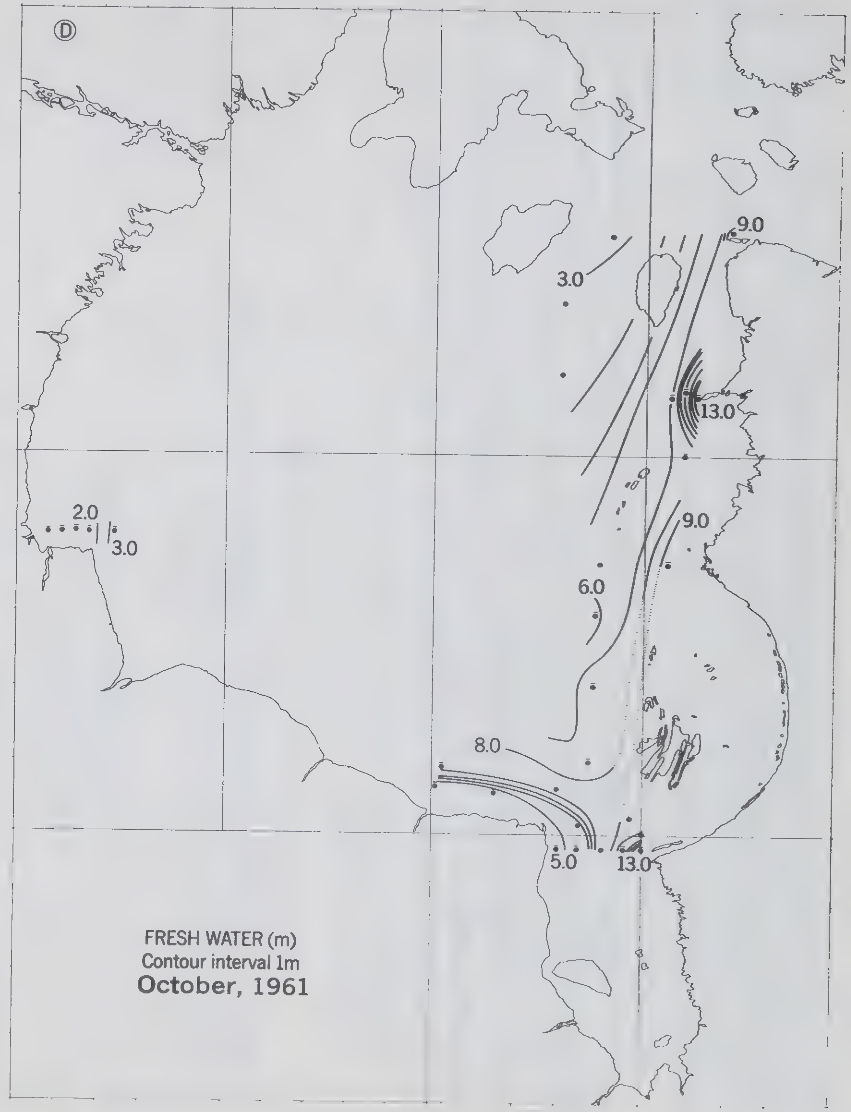


Figure 10. Distribution of the depth (m) of fresh water as derived in the text. The bar over the station position indicates that a salinity of 33.0°/00 was not attained at depth at the position. The dotted contour indicates that more than the usual doubt is attached the indicated configuration. (a) From

data observed about the latter part of July 1961. (b) From data observed in August 1961. (c) From data observed in September of 1961 and 1962. Stations at and south of latitude 59°N were occupied in 1961. (d) From data observed in early October 1961.





that about 1/3 to 1/2 of the precipitation would be so removed. The resulting annual drainage volume then appears to represent between 1/3 and 1/4 the total fresh water in Hudson Bay in August of 1961, or conversely the volume of fresh water in Hudson Bay represents an accumulation of drainage over an interval of 3 to 4 years.

During the winter a portion of the fresh water would be bound up in the ice phase, perhaps as much as that in a layer 1.5m thick over the region. An estimate of the amount of fresh water which exists during the period of ice cover based on the derived surface salinity of section 4.3 and an assumed mixed layer depth of 50m has been attempted. A tentative conclusion from this is that the amount of fresh water, including that existing as ice, does not show marked variation with season.

4.2 A volume budget

An evaluation of the net volume transport, based on the current distribution across the strait determined by Farquharson et al.(1960, their Figure 7), indicates a net outward flow of about $0.1 \times 10^6 m^3/sec$ was taking place. This would consist of the sum of the net flow through Fury and Hecla Strait (0.05 to $0.1 \times 10^6 m^3/sec$) and the fresh water derived from run-off. At the smaller estimated net transport through Fury and Hecla of $0.05 \times 10^6 m^3/sec$ the amount of fresh water would be $0.1 \times 10^6 m^3/sec$ less $0.05 \times 10^6 m^3/sec$ or $0.05 \times 10^6 m^3/sec$. A continuous discharge of fresh water out of the system at this rate is of the right magnitude necessary to remove the estimated $10^{12} m^3$ of fresh water entering each year from run-off. Using this value of the the net fresh water discharge and estimates of the salinity of the inflowing and outflowing water in Hudson Bay lead to an inflow estimate of $0.5 \times 10^6 m^3/sec$. Campbell (1958) estimated the inflow and obtained the generally similar values of $0.5 \times 10^6 m^3/sec$ for October, 1955 and $0.3 \times 10^6 m^3/sec$ for July, 1956. Combining these estimates suggests the water budget shown below:

Water Budget
($m^3 \times 10^6/sec$)

In		Out	
From the Atlantic through Hudson Strait	0.3 to 0.5	To the Atlantic through Hudson Strait	0.4 to 0.6
From the Arctic through Fury and Hecla Strait	0.05		
Fresh water from run-off	0.05		

The outflow through Hudson Strait is the sum of the inflows. The section across Hudson Strait where these values are considered to occur is located in the extreme west, for as described below it appears likely that a considerable portion of the water moving into eastern Hudson Strait may move out without entering Hudson Bay. The derivation from the presentation of Farquharson et al.(1960) of the value of the net volume transport in Hudson Strait included an evaluation of the outflow and

inflow in the section (Big Island to Wales Island); these were 1.0 and 0.9 x $10^{\overset{6}{\cancel{5}}}m^3/sec$ respectively. The values are larger, by nearly a factor of two, than those estimated for western Hudson Strait, and led to the consideration that a portion of the water entering Hudson Strait from the east changes direction within Hudson Strait. Such circulation has been described at the surface with regard to the movement of icebergs (Anon., 1954, p. 12) which are seldom seen further westward in Hudson Strait than Big Island. Apparently they enter from the north in the vicinity of Resolution Island, move northwest to Big Island, cross the strait to the south shore, and move eastward out of the region. Apart from the surface expression based on the movement of icebergs there does not appear to be other mention of this division of the inflow within Hudson Strait, although it was known that a significant portion of a southerly flowing current (Baffin Current) entered Hudson Strait (Smith et al.*, 1937). An important conclusion with regard to Hudson Bay is that variations with time of the transport in Hudson Strait, particularly in and east of the above section, may not necessarily reflect the influence of events occurring within the system.

In a detailed analysis of oceanographic data Campbell (1958) also attempted a determination of the net volume transport through Hudson Strait and obtained the following values,

Period	Net transport (x $10^6 m^3/sec$)
winter	0.1
July	0.3
late August - early September	less than 0.2
October	0.2

These range up to three times the value obtained above, i.e., three times 0.1 x $10^6 m^3/sec$. Also they suggest the existence of marked short-term effects. It appears that he considered that variations in the net discharge out of Hudson Strait in response to changes in the amount of fresh water entering Hudson Bay from run-off to be almost immediate, and remarked "It would be expected that the net volume discharge in Hudson Strait would reflect the rise and fall of the river run-off with a time delay of perhaps several weeks." Consideration of the estimates made here in the fresh water budget suggests that the effect of storage within Hudson Bay would tend to delay and smooth such responses. It may be that an over-estimate of the annual volume of fresh water input led to the conclusion. For example, he estimated that the volume of fresh water stored as ice represented "one-tenth of the total annual rainfall over the drainage basin", (p. 45), whereas the calculations here suggest that the annual run-off is very nearly equivalent to the fresh water stored as ice.

---

*The authors considered (p. 83) the volume inflow into Hudson Strait to be about 0.5 x $10^6 m^3/sec$.

4.3 The probable near-surface salinity distribution in winter

It was the conjecture of Hachey (1954) that "By freeze-up time the waters of Hudson Bay are quite uniform from top to bottom . . . . ". Dunbar (1958) discussed data from the northern part of Hudson Bay and considered that ". . . . there is no reason to suppose that the vertical exchange fails to reach the bottom of Hudson Bay." Thus both envisaged apparently, the complete removal of the summer stratification at some time during the period of ice cover. This would result presumably through the reduction of fresh water from run-off and through the removal of fresh water at the surface in the form of ice. Each factor would lead to an increase of the salinity in the near-surface such that the halocline observed over much of the area in summer would, by the end of the winter cooling period, not exist. The variation from a condition of marked stratification in summer to one of no stratification in winter would, if fact, constitute a major feature of the physical oceanography of the region.

Within Hudson Bay observations during the winter months are few. One series, taken from the ice in Coral Harbour and reported by Dunbar (1958), indicate a surface salinity of 33°/00. Comparison with data obtained there at "Theta" station number 163 in later summer (Anon., 1964, a) indicates that the value under the ice is the higher. In Foxe Basin the only winter data is that at Igloolik observed and reported by Grainger (1959). This series extended from September 1955 to September 1956 and has been interpreted as indicating that a variation of surface salinity of yearly period occurs with a maximum salinity of about 32.5°/00 in late winter prior to break-up.

In some areas it appears possible to infer a winter surface salinity from data obtained in the ice-free period in late summer. For example, in Omarolluk Sound in the Belcher Islands the sill depth is about 10m and the maximum depth about 200m. At "Theta" station 200 occupied there in September the salinity of the deep water was 29.9°/00. About this time the surface salinity in adjacent waters of Hudson Bay was observed to be as low as 25°/00. As the water at depth in Omarolluk Sound must have derived from near-surface water, it follows that a salinity variation occurs here also, and in the range of at least 25 to 30°/00 such that the higher salinity occurs in winter. Another example is that of Knight Harbour, Marble Island where a sill with a depth of about 6m and a maximum depth of 35m has been interpreted from information on Canadian Hydrographic Service chart number 5427. Consideration of "Calanus" station 7 (Anon., 1964, b) occupied there in July leads to the conclusion that a similar cycle occurs here as well with a range between the 30°/00 observed at the surface outside Knight Harbour and the 32.6°/00 observed at depth inside.

There are a number of other localities within the Hudson Bay - Foxe Basin system where limiting depths significant to this discussion may exist and where data in summer have been obtained. Examples include Richmond Gulf and Wager Bay where "Calanus" occupied respectively station number 38 in 1959 (Grainger, 1960) and station number 46 in 1961 (Anon., 1964, b). In the report concerning the 1959 work, Grainger indicated that a limiting depth to Richmond Gulf probably exists of about 13m. Examination of the station occupied in Richmond Gulf where a salinity of 27.3°/00 was observed at 120m suggests that a maximum surface salinity of at least this value is attained in winter. Information concerning the existence of a limiting depth to Wager

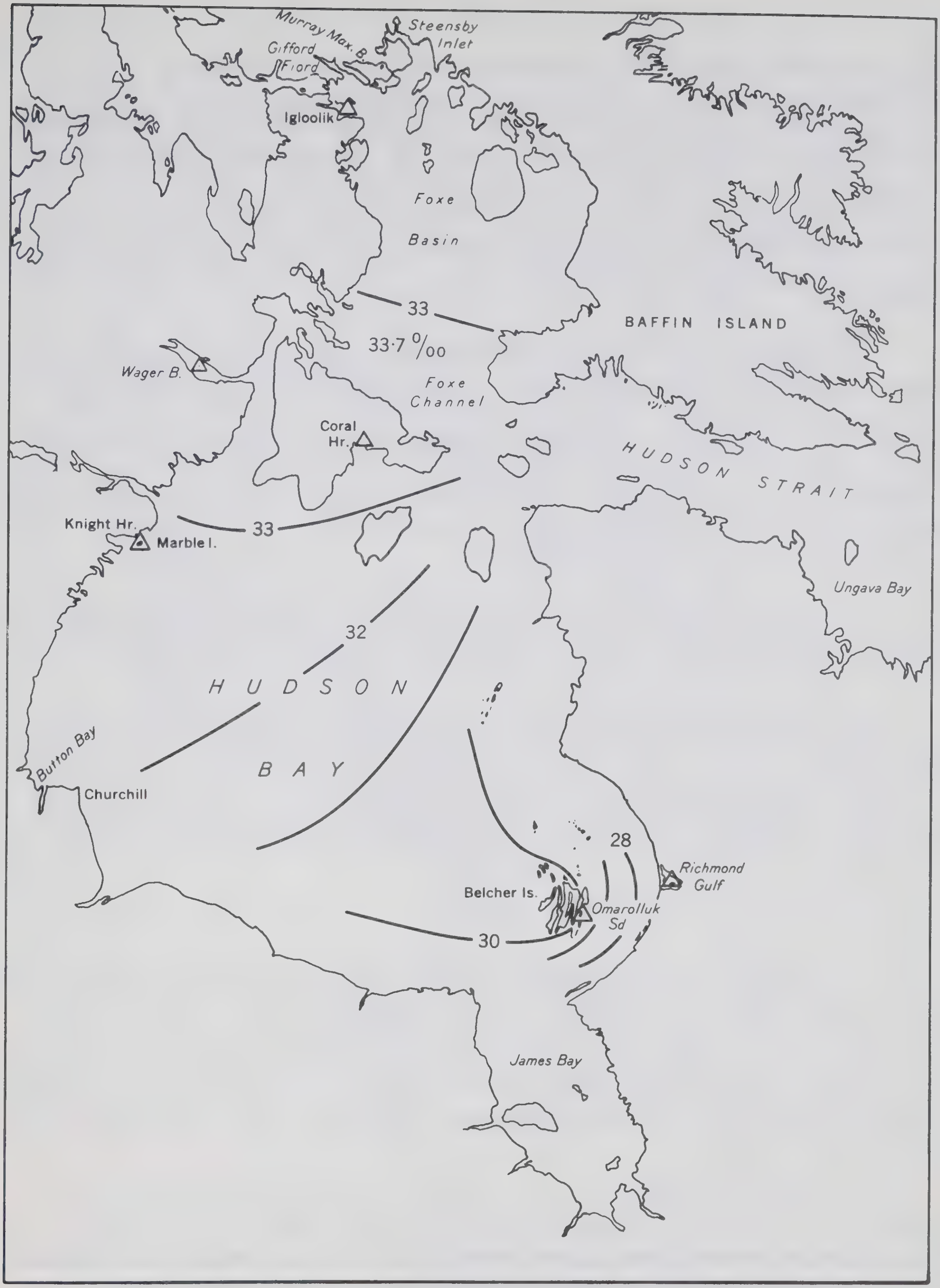


Figure 11. The probable distribution of winter surface salinity in the Hudson Bay - Foxe Basin system.

Bay is also not complete, but that available on the provisional edition of Canadian Hydrographic Service chart no. 5440 does not suggest one shallower than about 60m. An interpretation of the data obtained there suggests that the winter surface value is about 33.1°/00. Observations in Gifford Fiord and Murray Maxwell Bay obtained in "Calanus" (Dr. E. H. Grainger, personal communication) indicate that in both areas significant limiting depths likely exist and that the bottom salinity was about 32.5°/00. Consideration of the data described by Campbell (1964) concerning the high salinity water in Foxe Channel suggests that the winter surface salinity there may be as high as 33.8°/00.

A distribution of the surface salinity in winter derived from the foregoing is shown in Figure 11. Within Hudson Bay the distribution is similar to that which has been observed in August at 50m but indicates generally lower salinity values than have been observed at 75m and deeper. It is therefore concluded that although a yearly variation of salinity in the near surface with a maximum in winter does occur, a uniform condition from surface to bottom likely does not. Thus in this latter characteristic, the water of Hudson Bay is similar to that of other large water bodies of the world which have a seasonal ice cover.

Hudson Bay becomes very nearly completely ice covered (Lamont, 1949; Hare et al. 1949; Dunbar, 1954), so that it may be assumed that the surface temperature of the water at this time and in these areas is at the freezing point.

The relationship between the freezing point and salinity has been investigated by Thompson (1932) and Miyake (1939); the latter developed the formula which gives the approximate relation $T_f = 0.055S$, where T_f is the freezing point in °C and S the salinity in °/00. From this and the derived salinity distribution it may be seen that the surface water temperature in the ice covered areas under consideration would range from -1.85°C to -1.50°C.

4.4 The influence of ice cover in the 1961 season

It would be expected that a major feature of the heat budget of a surface water could be the extent and distribution of ice in the region during the summer. At this time in Hudson Bay, Dunbar (1954) noted that although it is "virturally clear even in the worst years", conditions are variable, and she provided an estimate of conditions at the "height" of extreme good and extreme bad seasons. Markham (1962) considered that usually within Hudson Bay the ice cleared from north to south as a result of "the long term mean wind flow".

It is the purpose of this note to remark on the apparent influence of ice conditions on the heat content of the surface water in the southwest part of Hudson Bay (Fig. 12) during the navigation season of 1961. Here ice persisted well into August by which time all other areas of Hudson Bay were free of ice. It seemed likely that the ice represented an accumulation, coincident with melting, due to a general movement into the region from the north caused by wind. In this way the surface water in the north might have experienced, as compared to that in the south, warming for a longer period; a circumstance which might be reflected in differing levels of heat

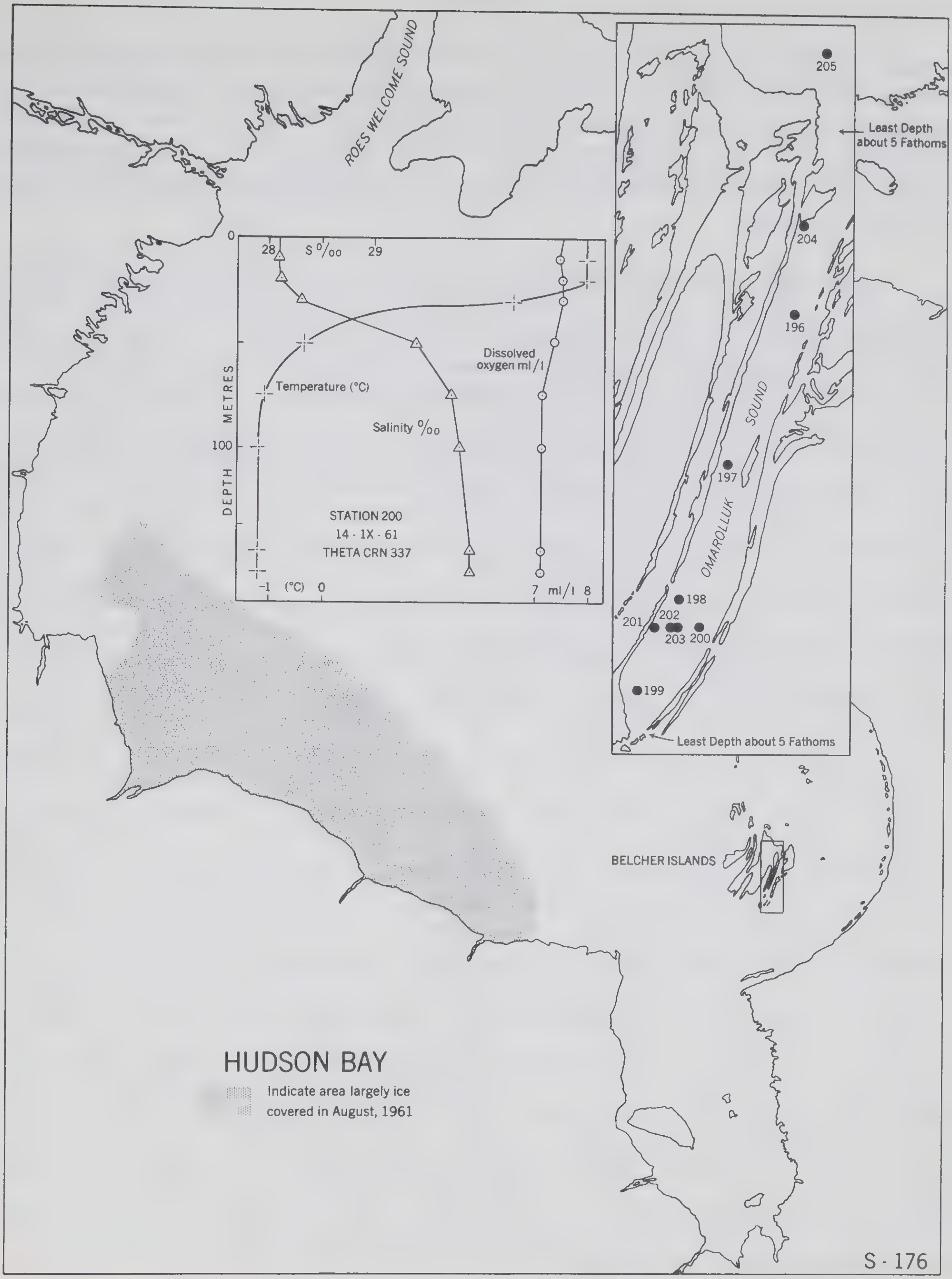


Figure 12. Location of Omarolluk Sound within Hudson Bay and the area in Hudson Bay where extensive ice cover persisted. Also indicated is the approximate position and number of the stations occupied in the sound, and the observed distribution of temperature (from BT and serial data), salinity, and dissolved oxygen at station 200 there on September 14.

storage. As the surface water temperature throughout the area at the end of the cooling season is nearly the same, i.e., at the freezing point, an origin is established for estimates of heat storage based on temperature observations at a later time. Such observations were made in "Calanus" and "Theta" (Anon., 1964, a and b) beginning July 22, 1961 throughout Hudson Bay including the aforementioned area of ice cover and in Omarolluk Sound.

Omarolluk Sound is about 40 miles long and 4 miles wide with a maximum depth of 180m and lies in a group of islands through which relatively shallow access to the adjacent waters of Hudson Bay exists in the north and south. In each access the sill depth as interpreted from available information including that on Canadian Hydrographic Service chart number 5470 titled Belcher Islands is estimated to be not greater than 5 fathoms. Serial data observed there in "Theta" on September 15 have been interpreted as indicating a seasonal heat storage of about 21 kilogram calories/cm^2. Observations about the same time in the region of ice cover indicate a heat storage of between 8-14 kilogram calories/cm^2. This difference in heat storage would represent, assuming the net of the radiative and flux terms to average about 250 gram calories/cm^2/July and August day, the influence of ice for a period of about 40 days longer in the southwest area than in Omarolluk Sound. The estimate requires a number of additional assumptions including such major ones as the similarity of climate in the areas and the insignificance of advection in Omarolluk Sound. Because of the restricted access the latter assumption does not appear unreasonable, but it does not appear possible to assess climatic influences which likely vary considerably over the extent of Hudson Bay. It is also likely that advection varies, and wind effect on movement would be more pronounced on ice than on water. Thus a deficit of heat in the surface water in one area would be compensated through an apparent surplus in another. An assessment of the heat storage to the latter part of July, 1961 in Hudson Bay north of the latitude of Churchill has been made (Fig. 13). Heat storage varied from less than 5 kilogram calories/cm^2 in the vicinity of areas of recent ice to values in excess of 15 kg-cal/cm^2 in the northeast and the northwest. Thus in much of the northwest the heat storage to the end of July was approaching that estimated for Omarolluk Sound by the end of the heating period, i.e., by September 15, so that it appears that this area of Hudson Bay cleared of ice earlier than Omarolluk Sound. Interpretation (Fig. 14) by Archibald et al. (1962) of ice observations from aircraft indicated that considerable open water existed in the northwest by June 21 at least and to the northeast by the end of June at least. The relatively high heat storage indicated (Fig. 13) in these areas to the end of July does not therefore appear unreasonable. The combination of all the data suggests that part of this ice moved from the north into the southwest, as described by Markham (1962), to accumulate and finally to melt there. The consequence of an ice accumulation then would be reflected not only in the distribution of temperature and heat content but also in the distribution of salinity.

An area of relatively high surface salinity (32°/00) within Hudson Bay during the 1961 season was that of Roes Welcome Sound. There the heat storage to August 10 was such (Fig. 13) as to be comparable to that described for north Hudson Bay. It seems, therefore, that Roes Welcome Sound also must have cleared of local ice relatively early, and not to have filled subsequently with drift ice from the north as suggested in the pilot (Anon., 1954). The 1961 experience is also contrary to that described by Campbell

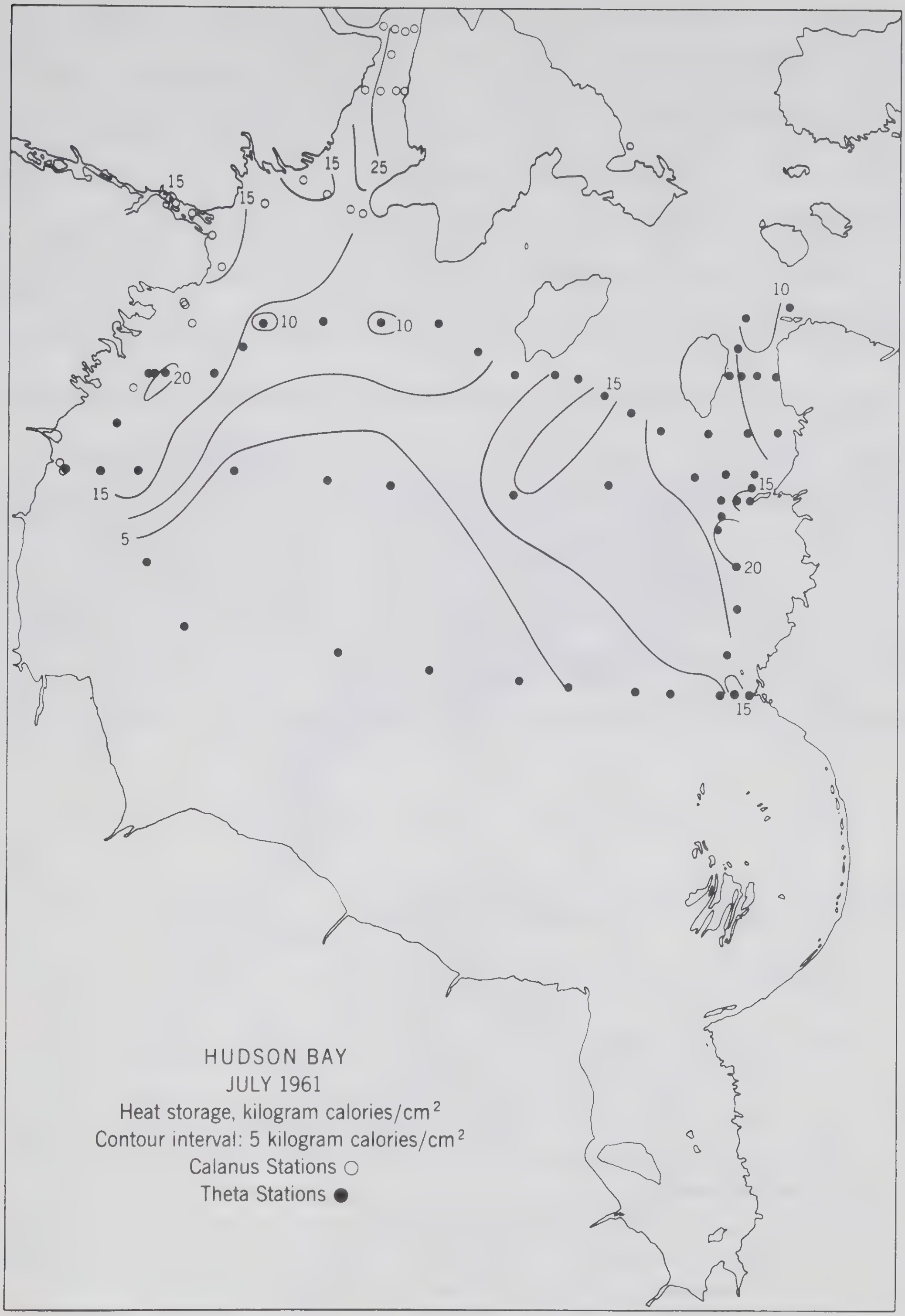


Figure 13. Estimated heat storage in kilogram calories per square centimeter in period July 22 to 31, 1961 based on serial and BT data of "Calanus" and "Theta". In the area of Roes Welcome Sound the "Calanus" observations extended to August 10. The origin for the estimate was -1.0°C.

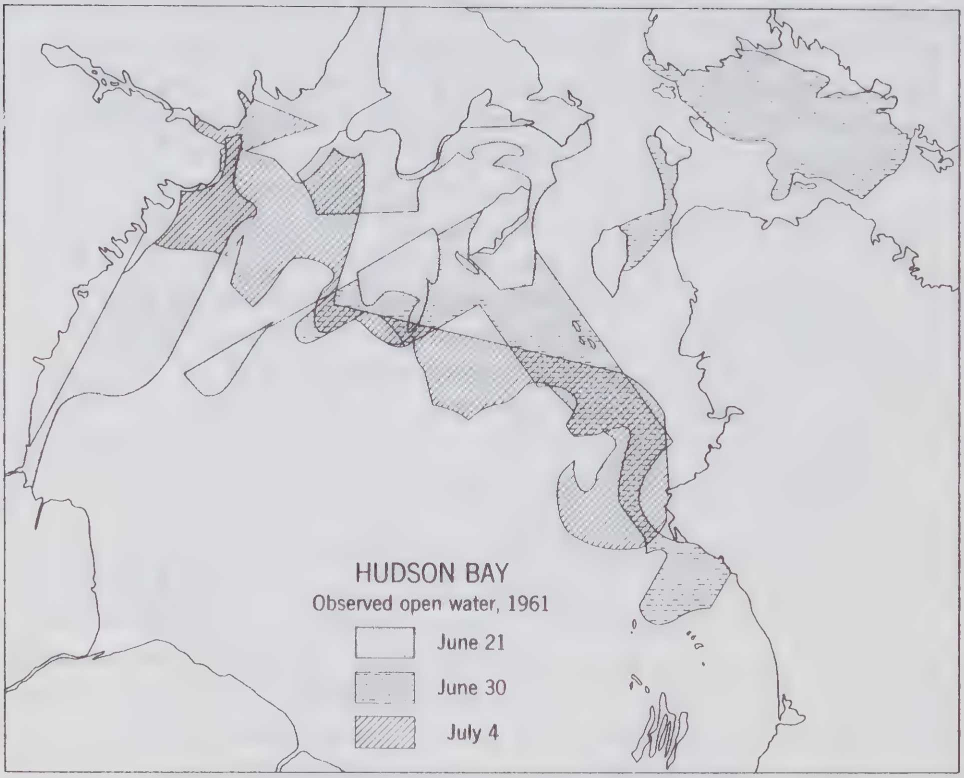


Figure 14. Observed areas of open water in northern Hudson Bay of June 21, June 30, and July 4 from the interpretation of Archibald et al.(1962) based on limited air survey.

(1958) who determined that a net transport of $0.15 \times 10^6 m^3/sec$ occurred through Roes Welcome Sound from Foxe Basin into Hudson Bay in each of the seasons of 1955 and 1956. Such a transport would require a current of about 30cm/sec throughout the most restricted section (depth 53m, area $0.564 \times 10^6 m^2$) of Roes Welcome Sound, which would have brought a colder water from Foxe Basin, and perhaps ice. That it did not bring ice south is of interest, for it appears that generally in the 1961 season and after the initial break-up condition was established, ice did not enter Hudson Bay from the north. Should the export of ice out of Hudson Bay into Hudson Strait prove to have been not less than average, the 1961 season would likely rate as a good year in the scale of Dunbar (1954). In consequence the heat storage in the waters of Hudson Bay at the end of the heating season of that year may have been greater than normal.

4.5 A preliminary heat budget of the sea in the vicinity of Churchill for 1961

This examination of features of the heat budget in the vicinity of Churchill for 1961 has been carried out as part of the study of oceanographic data obtained in Hudson Bay during the navigation season of that year. It appears that no similar study has been made, perhaps because of the lack of suitable data. It is to be recognized that paucity of data remains a problem, particularly for the period that Hudson Bay is generally non-navigable, so that the results obtained here for the region are preliminary only. The work constitutes an attempt to apply available formulation to oceanographic and meteorological data in order to assess individual terms of the heat budget. This may be expressed, after Sverdrup et al.(1942), for a column of water in the sea during any interval Δt as,

$$Q_s - Q_r - Q_b - Q_e - Q_h + Q_v - Q_\theta = 0 \qquad (1)$$

where

Q_s is the shortwave solar radiation both direct and diffuse incident on a horizontal surface at sea level,
Q_r is the shortwave radiation reflected from the sea surface,
Q_b is the effective longwave back radiation,
Q_e is the heat evolved through evaporation (sublimation of ice),
Q_h is the sensible heat conducted from sea to atmosphere,
Q_v is the advective heat brought into the area, and
Q_θ is the heat storage in the column.

All other sources of heat such as conduction through the sea floor, biological and chemical activity, and tidal and wave energy are assumed to be several orders of magnitude less than solar radiation and, therefore, of no significance to this energy budget. The units used are gram-calories per square centimeter per 24 hours ($g\text{-}cal/cm^2/day$).

Values of sea surface temperature in the absence of ice were obtained from observations in "Theta" (Anon., 1964, a) in Hudson Bay in 1961 and through interpolation of the temperatures likely existing at freeze-over and break-up. During the winter months it was assumed that the snow and ice surface attained the temperature of the air immediately above. The extent of ice cover during break-up and the date of

Table III. Calculated mid-monthly values of the absorbed radiation (g-cal/cm^2/day) in the vicinity of Churchill (58°45N94°04'W) and related factors.

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Q_{os}	90	180	350	560	730	780	730	590	410	230	100	50
C	5.1	4.4	6.9	7.3	7.2	7.1	7.4	7.8	8.2	8.3	8.0	7.4
Q_s	80	167	258	386	512	557	493	366	229	125	59	34
A_n	10.3	18.8	29.3	41.2	50.3	54.7	52.8	45.3	34.3	22.8	12.8	8.1
$\bar{h}$	5.6	10.2	15.8	21.8	27.2	29.6	28.6	24.5	18.5	12.1	6.9	4.4
r	.75	.75	.75	.70	.50	.45	.17	.11	.13	.18	.55	.75
Q_r	60	125	194	270	256	251	84	40	30	23	33	25
Q_o	20	42	64	116	256	306	409	326	199	102	26	9

Q_{os} = the mean solar radiation (g-cal/cm^2/day) incident on the sea surface under a clear sky (Mateer, 1955).

C = the daylight cloud cover in tenths of sky covered.

Q_s = the estimated solar radiation (g-cal/cm^2/day) incident on the sea surface from formula 2 of the text.

A_n = the solar noon altitude (degrees).

$\bar{h}$ = the mean solar altitude (degrees).

r = the fraction of the solar radiation reflected from the surface as estimated in text.

Q_r = the mean value of the solar radiation (g-cal/cm^2/day) reflected from the sea surface.

Q_o = the absorbed solar radiation (g-cal/cm^2/day).

freeze-over were estimated from the report of Archibald et al.(1962) based on air reconnaissance. Meteorological data for 1961 at Churchill were obtained from reports of the Meteorological Branch (Anon., undated a, b). During the period of ice and snow cover it was assumed that conditions over the sea would be the same as over land, i.e., the same as at Churchill. For the other months a correction based on observations in "Theta" was applied to the Churchill data.

4.51 Short-wave radiation from sun and sky

The mean solar radiation under clear skies, Q_{os}, was interpolated from monthly charts prepared by Mateer (1955, a) and the mean monthly day-light cloud cover, C, obtained from meteorological records (Anon., undated a), which also include information concerning observed solar radiation at Churchill for the period January 1959 to June 1961. An estimate of the average monthly insolation received at the surface, Q_s, over Canada was provided by Mateer (1955, b) which included an empirical equation relating average insolation to cloudiness. A modification of this formula, and similar also to that derived by Laevastu (1960), of the form

$$Q_s = Q_{os}\,(1-0.0008C^3) \qquad (2)$$

was applied to obtain the tabulated values of Q_s of Table III.

4.52 Reflected short-wave radiation

The short-wave radiation absorbed at the surface, Q_o, shown in Table III was obtained from Q_s after a value of the albedo, r, was determined for each mid-month. The albedo is primarily a function of the solar altitude and the nature of the surface. To a lesser degree it is affected by the cloud cover and the height, and also the turbidity of the air mass. Anderson (1952) described the albedo of a lake surface as,

$$r = a\,\bar{h}^{\,b} \qquad (3)$$

where $\bar{h}$ is the mean solar altitude and a and b are empirically defined constants dependent on cloud height and cover. The average cloud cover was broken middle cloud for most of the year. Although Anderson does not give values for middle cloud, a linear interpolation gave values of $a = 2.18$, $b = -0.97$ for middle scattered, and $a = 0.96$, $b = 0.68$ for middle broken. Values of r for August to October were calculated accordingly and entered into Table III. For other months use was made of the work of Williams (1961) and his estimates of the albedo of snow and ice surfaces as follows,

Surface	r (albedo)
New Snow	.80 - .90
Old Snow	.60 - .80
Melting Snow	.40 - .60
Ice	.40 - .50

During 1961 snow fell on practically every day during the winter months. This should result in an average new snow cover. However, the winds are quite strong in this region and, therefore, it is possible that considerable drifting took place leaving bare patches of ice and old snow. In consideration of this, an albedo of 75% was taken for the winter months. As little sunlight occurs during this time of the year, a large error in r will not seriously affect the yearly totals. From ice reports during the spring of 1961 it was estimated that the melting of snow began toward the end of April and into May. As snow was falling nearly every day at this time, the combined effect of fresh and melting snow is estimated to have produced an albedo of 70%. During May, an ice cover of 9/10 was reported with no puddles. Assuming some melting snow cover, then the albedo was approximately 50%. Reports for June indicated 8/10 ice cover, 2/10 puddle cover, hence on albedo of 45%. Average ice conditions in July were 3/10 ice cover with puddles and 7/10 open water, hence an albedo of 17%. Open water then occurred until freeze-over on the middle of November. November conditions were estimated to be 50% ice cover with new snow, so that an albedo of approximately 55% was assigned.

4.53 Effective long-wave back radiation

The sea surface emits long-wave radiation energy in proportion to the fourth power of its absolute temperature. The wavelength of this radiation is also determined by this temperature. The wavelengths emitted by the sea surface are almost completely absorbed by the atmosphere (Greenhouse effect), which also radiates back to the sea surface. The remainder is the effective long-wave back radiation Q_b. This is tabulated in Table IV as calculated using formulation derived by Anderson (1952) in the Lake Hefner study. It was assumed that the equations obtained at Lake Hefner would apply to the Churchill area, and it was expected that errors introduced by this assumption would be small compared to errors in cloud cover and cloud height observations. It was also assumed that ice surfaces and snow surfaces have the same emmissivity factor as water surfaces. It is known that snow is an efficient radiator so that its emmissivity factor is probably approximately equal to that of water. However, ice cover becomes a less efficient radiator as more air bubles become trapped in it. During the spring this might be a source of error, but it is possible that a thin layer of melt water is responsible for the radiation during this time.

It is noted that the values of Q_b were calculated assuming the existence of a complete ice cover during the winter months.

4.54 Evaporation

The calculated values of the energy exchange associated with evaporation and sublimation, Q_e, are tabulated in Table IV. During much of the year and especially in late winter Q_e has little effect on the energy balance. Most evaporation occurs from August to November; the total for the year was about 0.2m.

Table IV. Mid-month values of each of the radiative and flux terms, the precipitation falling as snow, and the open water correction, all in g-cal/cm^2/day.

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Q_o	20	42	64	116	256	306	409	326	199	102	26	9
Q_b	87	96	93	103	112	122	100	80	77	116	166	77
Q_e	4	7	10	17	28	20	23	62	74	90	99	6
R	0.0	0.0	0.0	0.0	.23	.38	-70	.31	.32	1.3	1.9	0.0
Q_h	0	0	0	0	6	7	-16	19	24	112	189	0
P	-5	-5	-10	-17	-6				-8	-14	-23	-6
c	-19	-16	-12	-7								-15
net	-90	-78	-51	-11	109	157	302	165	24	-216	-428	-89

Q_o = absorbed solar radiation from Table III.

Q_b = effective back radiation.

Q_e = evaporation.

R = Bowen Ratio.

Q_h = sensible heat conduction.

P = heat required to melt the precipitation falling as snow.

c = correction for open water from Table V (i).

net = $Q_o - Q_b - Q_e - Q_h + c$.

4.55 Conduction of sensible heat

The amount of sensible heat, Q_h, conducted away from the sea by the atmosphere is dependent on the vertical temperature gradient and the eddy conductivity. The eddy conductivity and the amount of sensible heat conducted are both difficult to measure directly. If it is assumed that the co-efficient of eddy conductivity of water vapour is

identical to the co-efficient of eddy conductivity of heat, then as first suggested by Bowen (1926),

$$\frac{Q_h}{Q_e} = R = 0.61 \frac{(T_s - T_a)}{e_s - e_a} \tag{4}$$

where T_s and T_a are the sea and air temperature (°C), and e_s and e_a are the respective vapour pressures (mb). Values of (4), the Bowen Ratio, and of Q_h are entered in Table IV. It appears that sensible heat conduction is significant in the autumn and was an important source of atmospheric heat at that time.

4.56 Precipitation

The energy exchange which could be associated with the melting of that precipitation falling as snow is shown in Tables IV and V in the months that it occurred. As the ultimate fate of the snow is not known the value is not included in the net of Table IV, but is included in the net of Table V, and in the estimate of the change of heat storage of Table VI.

4.57 Net of the radiative and flux terms

Were it not for the existence of open water during the winter months the net of the radiative and flux terms at mid-month could be obtained as the sum of $Q_o - Q_e - Q_b - Q_h$. The open water necessitates a correction based on the amount of open water. When the wind blows steadily from the south or southwest for several days at Churchill a lead about 10 miles wide opens between fast ice and pack ice. This lead closes if the wind changes its direction to north (Donovan, 1957). Assuming that this lead represents the sum of all the leads to the north, then the area of open water is 2% of the total area. With a surface water temperature of -1.6°C during the period of ice cover, the several terms of the heat budget for open water were calculated (Table V). In each of these months except May a net loss of heat is indicated. The loss would occur mainly through conduction of sensible heat and effective back radiation, and to a lesser extent through evaporation.

Each mid-month net value of Table IV includes the correction for the existence of 2% open water. The greatest gain occurred in July and the greatest loss in November; the periods of no-net-transfer occurred in April and September. The yearly gain of heat was 22,690 g-cal/cm^2, and the yearly loss was 23,405 g-cal/cm^2. Due to the difficulty of estimating an average condition for early November it is considered that the flux of heat to November 15 could be substantially in error.

Another factor which could influence significantly the annual totals is the value estimated for the albedo during the break-up in June and July. In these months in the region seaward of Churchill extensive open water developed to the north with an accumulation of ice to the south. The ice appears to have consisted to considerable extent of an ice advected from the north as described in section 4.4. The ice would not only reduce the value of the adsorbed short-wave radiation through increased albedo (of

Table V. (i) Mid-month values of each of the radiative and flux terms over open water, precipitation included. The terms are as in Tables III and IV; the net is the sum of $Q_o - Q_b - Q_e - Q_h + P$.

	Dec	Jan	Feb	Mar	Apr	May
Q_s	34	80	167	258	386	512
r	.35	.30	.20	.15	.12	.10
Q_o	22	56	134	221	342	461
Q_b	246	295	279	248	185	110
Q_e	150	160	170	164	148	46
Q_h	383	503	484	406	354	12
P	-6	-5	-5	-10	-17	-6
net	-765	-907	-804	-607	-362	-287

(ii) Mid-month values of ice thickness (cm) for December 1960 to May 1961 as interpreted from Swanson (1961) and Anon. (1961), and the heat, Q_L, released by ice information.

	Dec	Jan	Feb	Mar	Apr	May
ice	50.8	86.4	108.8	123.0	134.7	134.7
Q_L	-112	-92	-57	-37	-30	0

the ice compared to open water), but also would reduce the sensible heat by the amount required to change the temperature and melt the imported ice. An indication of the influence of this factor on the seasonal heat storage in this area during August is shown in Figure 18. In this a marked gradient existed seaward of Churchill, there being much more heat to the north than to the south. Here the ice persisted well into August so that it is considered that the greatest gain (the net of the radiative and flux terms) probably occurred about this time.

Because of the uncertainty attached the estimate of each term of the budget neither the net loss nor the net gain can be considered precise. Similarly it would not be realistic to attach too much significance to the deficit (sum of net loss and net gain) of 700 g-cal/cm^2, although it may be generally representative of ice covered regions within Hudson Bay. If so the deficit must be balanced* by advection either

*For an individual year the deficit may not be balanced but on the average must be or the region would become colder.

Table VI. Mid-month values of the latent and sensible heat required to change the state and temperature as described in the text, and the sum of the net and the heat storage, Q_θ.

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
net	-90	-78	-51	-11	109	157	302	165	24	-216	-428	-89
Q_θ	-"97"	-"62"	-"47"	-"47"				250	230	-400	-400	"-118"
Q_v								85	206	184	28	

net = as obtained for Table IV.

Q_θ = heat required to change the state and the temperature. Values in italics represent latent heat and are from Table V (i) and (ii), and are = $P + Q_L$.

Q_v = heat brought into a region by currents and here determined as the remainder of the net and the sensible heat storage.

through a net export of ice out of Hudson Bay or by the import of relatively warm water. Data concerning the actual or relative amounts of imported and exported ice are not available so that this interesting aspect cannot be examined. The estimated upward movement of 0.4×10^{-4}cm/sec (Barber, in review), at an average temperature difference between the warmer deep water and the surface water of 0.3C° would provide about half (380 g-cal/cm^2) the deficit. This movement is associated with the seaward displacement within the mixed layer of the fresh water entering the system from run-off.

4.58 Heat storage and advection

That a seasonal variation in the value of the heat storage, Q_θ, occurs is suggested by the observation that in winter an ice cover forms and in summer the ice is dissipated and a thermocline develops. The data are not sufficient to allow an assessment of the heat storage at each mid-monthly interval but a number of values have been derived and entered in Table VI. For the months of ice formation (December to April) the latent heat released by the formation of ice (Table V) is considered the major loss, but one which need not be balanced by advection. During the period of ice (and snow) melt the storage term includes the latent heat but it does not appear possible to estimate the appropriate mid-month values. August and September values were obtained from the temperature data observed seaward of Churchill at "Theta" stations 63 (August 3; 5.0 kg-cal/cm^2), 192 (September 11; 14.9 kg-cal/cm^2), and 240 (September 28; 18.8 kg-cal/cm^2). In October and November the sensible heat stored during the ice free period is given up mainly through back radiation and convection. The assumption of complete freeze-over on November 15 at which time the surface temperature would be at the freezing point and the surface layer about 40m

allowed an estimate of the heat loss from the time of the occupation of station 240, hence the mid-month values.

Values at each mid-month of the advected heat, Q_V, as the difference between the net of the radiative and flux terms and the sensible heat storage were possible for only the months of August to November (Table VI). In each of these months heat was advected into the area with a peak in September. From the knowledge of the distribution of heat at this time it would seem that the heat existing to the north was advected into the area off Churchill by a southerly directed water movement. During the ice-free period and during break-up the advection of water and ice from a general northerly direction can be a significant factor in the summer heat budget of the water in the vicinity of Churchill.

4.59 Other aspects

Of interest in the assessment of the heat budget in ocean areas is the extent to which deductions concerning water movement follow. The present work cannot be considered particularly informative in this respect except as already mentioned. One reason for this is that after the formation of an ice cover the water temperature in the mixed layer is at the freezing point so that a horizontal temperature gradient does not exist, or is very small. Horizontal water movement within the layer then would not advect significant amounts of heat. This consideration is distinct from that relative to the heat flux due to entrainment into the mixed layer of deeper water. As was calculated the latter process could provide an amount of heat equal to about half the estimated deficit.

The remaining portion of the deficit could be met by a net export of ice, and in the absence of data one may speculate that it may be a representative value for the net export. If it is, it would suggest that Hudson Bay does not contribute greatly to an ice movement to seaward out of Hudson Strait*.

Another process which might provide additional heat to meet the deficit is one related to the growth of the mixed layer whereby a depth increase of the layer may occur during the period of active ice formation. This feature has been mentioned but it is emphasized that data concerning this in Hudson Bay are not availabe. Coachman (1963) indicated that in the Arctic, summer influences penetrated to a depth of about 25m and winter processes deeper but not deeper than 50m. The increased depth in winter was associated with an increase of salinity in the layer apparently due to the increase of density at the surface and the consequent vertical mixing. By analogy with the evolution of the mixed layer in the absence of ice, it seems that the ultimate

*Smith (1931, p. 49 to 51) reviewed information concerning the movement of ice within the system and provided an assessment of the relative contribution of the various sources of ice to the "pack to Labrador". Apparently he did not consider the contribution from Hudson Bay to be particularly significant, but it is to be noted that on the basis of information then available he was able to remark "The bay itself remains comparatively free from ice during winter except for a 5 to 6 mile wide fringe".

depth of the winter mixed layer is determined more by vigour of the mixing process there and less by the depth within the halocline that the salinity of the mixed layer is attained. Thus the increased depth of the mixed layer would involve a change in the structure of the top of the halocline and a flux of salt into the layer. This salt would contribute to the overall salt balance, the consideration of which led to the derivation of the value of the average upward movement into the mixed layer and the heat contributed to the layer from the deep water. It is concluded therefore that the flux of heat which may be associated with the change of structure of the mixed layer in winter cannot be considered as being additional to that already derived.

Another aspect of the possible deepening of the mixed layer relates to the existence of supercooled water. From the foregoing it follows that two processes likely exist which lead to a flux of salt into the mixed layer during the winter. One occurs because of the freezing out of fresh water at the surface and the formation of an instability there; the other results as a secondary effect of this formation. Untersteiner et al. (1964) assumed a model of the mixed layer in which the only salinity change allowed was that related to the freezing out of fresh water. The existence of another salt flux mechanism suggests the model may not be appropriate, and hence the conclusion concerning the existence of "substantial super-cooling" need not follow, even though it may be fact.

4.6 A model

It does not appear possible to ascribe a degree of importance or relative significance to any one of the various factors which contribute to the character of Hudson Bay water, although the free connection to the Atlantic Ocean, the large run-off, and tidal mixing in Hudson Strait would appear to be among the first in importance.

The first makes it possible for a water close to oceanic salinity to occur in Hudson Bay, the second likely drives the main circulation, and the last leads to modification of the water to such an extent that it may be considered peculiar to the region. Elaboration beyond this is difficult, and becomes somewhat speculative. This will be apparent in this section where an attempt is made to bring together all that is available in order to provide a simple model of events, and to include something of the circulation within Hudson Bay. Little direct evidence of the movement of water exists but this, together with inferences derived as in the following paragraph have provided the attempt with a measure of success, it is believed.

As suggested in the description of the time series data of Figure 6, a trend to a water warmer and containing more oxygen is indicated in the north late in the season. The high oxygen values indicate that the water comes from Hudson Strait and is not entirely a re-circulation of Hudson Bay water. That such a warm water could exist within Hudson Bay was indicated earlier in the data presented by Bailey et al.(1951), wherein a water in the north central area at 100m about -1.2°C and 33.0‰ was observed during the period September 9-16, 1948. Primarily because the deep water salinities of 1948 were greater than those of the 1930 "Loubyrne" observations, they considered that the feature was due to an "increasing Atlantic influence" associated apparently with an amelioration of the general climate (Dunbar, 1958). Interpretation

of the 1961 and 1962 data suggest it probable that the explanation, when known, will include the influence of seasonal effects, and in the following discussion this will be emphasized. However, it is important to recognize that the extreme of the seasonal trend to a warmer deep water in Hudson Bay may not yet have been observed, so that it is not possible to state with confidence the extent to which average or steady state conditions are represented in the usual late summer (August and September) distributions. Consideration of the extent to which seasonal changes of temperature and salinity have been observed in Hudson Strait tends to support the view that further change could occur. Bailey et al.(1951) were not aware of the occurrence of such a seasonal cycle and therefore were not able to discuss it in relation to a longer term Atlantic influence. Available data are not sufficient to provide for rejection* of their hypothesis. There are therefore two aspects of the observed changes which must be considered; these are that,

(1) the more recent, 1948 and later, observations were made during a period of relatively strong Atlantic influence, and

(2) a cyclic variation with yearly period occurs in the deep water in the northern part of Hudson Bay.

Dunbar (1951; 1958) envisaged a direct relation between variations in an Atlantic influence and climatic changes. He compared (1958, p. 186) the temperature-salinity relationship of the 1953 and 1954 "Calanus" observations with those of "Loubyrne" and "Haida" and concluded that there was no evidence in the "Calanus" material to support the existence of climatic change since 1948. Beyond this the first suggestion above does not appear to have been examined further except here in a footnote in which doubt is cast on the basis for the argument that climatic change had occurred in the interval including 1930 to the present. The second aspect is of interest for it leads to consideration of the nature of the seasonal cycle of the deep water, and in particular, of the process which determines the cold portion of the cycle. Campbell (1964) and Barber (in review) described the process each considered significant. While quite different the processes are similar in that they proceed in winter and occur primarily at the surface in association with cooling and ice formation. It seems that a quantitative assessment of the heat and fresh water budgets would allow definitive statement relative to the significance of either process, although sufficient data may not exist. That an extreme cold and deep water exists in Foxe Basin and northern Hudson Bay appears fact nevertheless, so that the subsequent fate of this water within Hudson Bay is of interest as well as its participation in the observed seasonal cycle. It is proposed

---

*In the main the physical evidence for "long-term" changes of the temperature of the water in Hudson Bay appears to be tenuous for,

(1) the "Acadia" data of the 1930 season do not support the hypothesis based on the "Loubyrne" 1930 data that a particularly cold water occurred in Hudson Bay in that year, and

(2) while the "Acadia" data of 1929, 1930, and 1931, (Fig. 15) do not indicate the existence of the tongue-like distribution observed in 1961 and 1962, the general temperature level in the deep water does not appear significantly colder than that observed in August, 1961.

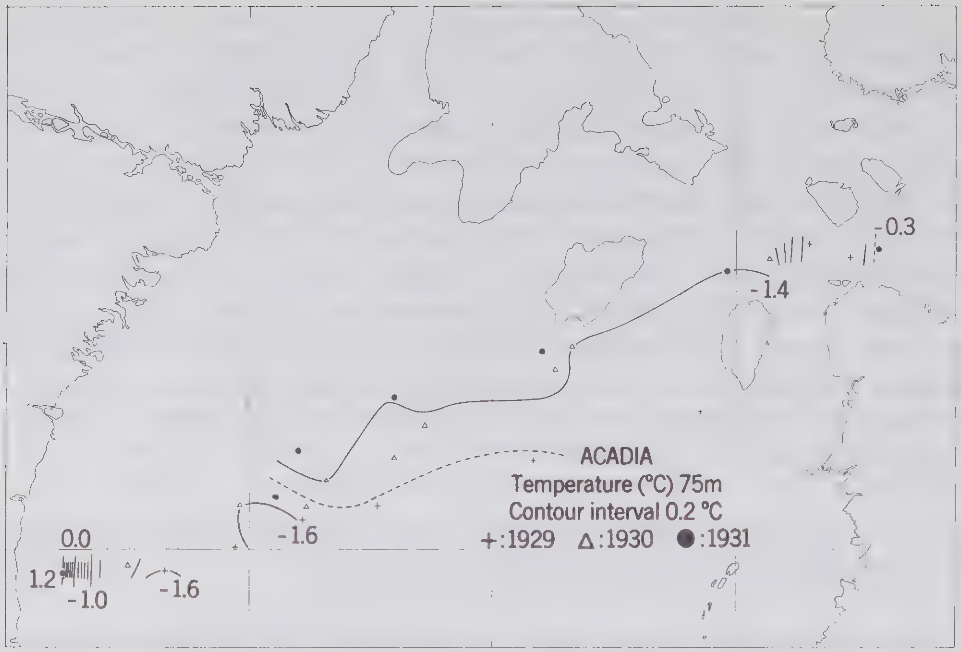


Figure 15. Temperature (°C) distribution at a depth of 75m from data observed in "Acadia" in 1929, 1930, and 1931.

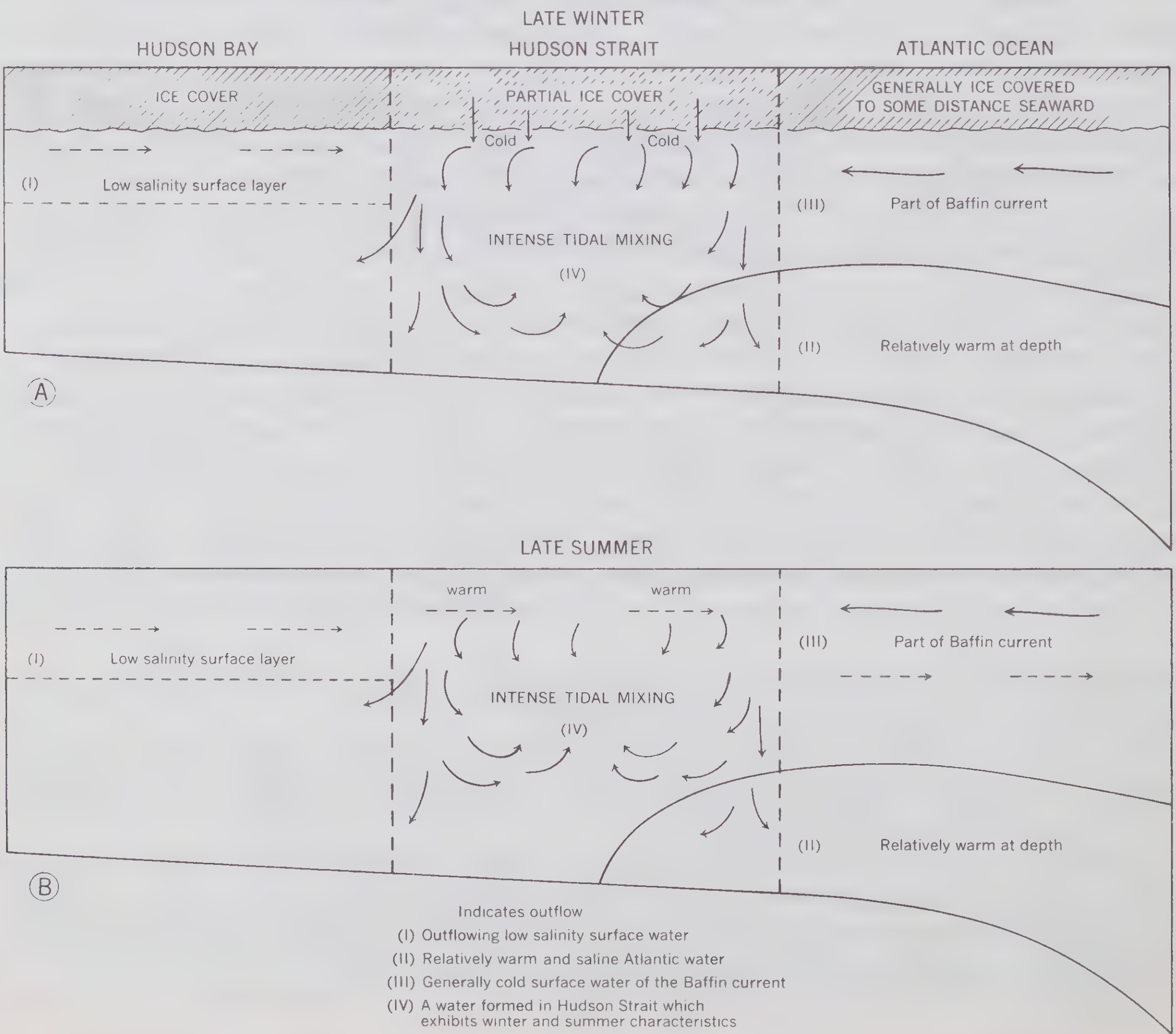


Figure 16. Schematic presentation of (a) the late winter condition, and (b) the late summer condition.

to examine this through use of a schematization (Fig. 16) or model of the region. In this it is assumed that the influence in the system of any transport through Fury and Hecla Strait is not significant. Two situations are considered, one in late winter the other in late summer. In both a low salinity layer is considered to exist over most of Hudson Bay but in the winter situation it is considered that over much of Hudson Strait and northern Hudson Bay an isohaline condition is approached. Mixing due to strong tidal currents is considered to occur generally throughout Hudson Strait to Roes Welcome Sound, and particularly in and adjacent to the north-south section containing Digges, Nottingham, Salisbury, and Mill Islands. The quantitative significance in the model of the mixing due to tidal activity has not been assessed, but it appears that at least in the aforementioned section, the structure is very nearly isohaline except close in to the Digges Island side due to the large volume of outflowing low salinity water from Hudson Bay. It may be emphasized that while the role of tidal activity is likely not critical to the later winter model as here presented, it probably plays a further role through both mixing and advective processes which maintain significant ice free areas in winter.

As depicted in the schematized late winter situation a partial open water condition is indicated in the vicinity of Hudson Strait (this may on the average have a particular pattern not now known) where radiative and flux processes proceed unrestricted (by ice cover) and lead to extensive cooling. The water thus cooled is relatively dense and initiates vertical movements, likely to the bottom in some areas, and moves away from the region at depth into Hudson Bay and into the ocean. It is believed that the formation of the water may be enhanced through the movement into Hudson Strait at the surface of a water recently influenced by cooling and freezing along the east coast of Baffin Island. The contribution into Hudson Bay must mix to some degree with the deep water there but as shown a portion retains its identity at least into the following summer. With the advent of summer the process declines and by late summer the ice cover is almost completely removed and the low salinity surface layer extensive and well-defined. The seaward movement of mixed fresh and salt water of this layer is compensated through a shoreward movement of the deep water. Some mixing of these waters in the area of strong tidal mixing occurs so that the deep shoreward movement consists in part of surface water within the system and an oceanic water. The first may be quite warm or contain considerable heat as may the water of oceanic origin so that the shoreward movement is relatively warm. Thus the seasonal cycle in the deep water is maintained by the continuing influence of an Atlantic water, the formation of a cold water at the surface in winter, and the downward mixing of heat absored at the surface in summer.

The extent to which the seasonal cycle is observed at depth within Hudson Bay is determined by the cnaracter of the pattern of water circulation. The relation of the main feature of the pattern to the estuarine circulation has been described, and the Pilot (Anon., 1954), Dunbar (1958), and Campbell (1959) provided information concerning currents in the northern approach passages. Except for the strong outflow from Hudson Bay in the vicinity of Digges Island the advice is somewhat contradictory, perhaps because direct evidence of features of the water movement within Hudson Bay is meagre. Remarks based on casual observations of the drift of flotsam and jetsam (Anon. 1954 p. 10;273) Flaherty, 1918, p. 458) indicated a movement from west to east across southern Hudson

Bay and to the north out of James Bay and along the eastern side into Hudson Strait. Along the eastern side shoreward of the Ottawa Islands the movement is described on Canadian Hydrographic Service chart 5449 as "strong northwest current" and "constant northerly current". South of the Ottawa Islands to James Bay some detail of the current pattern based on an interpretation of temperature and salinity observations was provided by Grainger (1960). At the surface and 10m a general movement occurred out of James Bay northward to the east of the Belcher and Ottawa Islands, but with a significant movement to the northwest out of James Bay passing southwest of the Belcher Islands. At 50m the latter movement was pronounced and appeared part of a general clockwise circulation around the Belcher Islands at the depth.

During the 1961 season an attempt to follow the drift of the Rankin Inlet navigation buoy was not conclusive. Apparently the buoy went adrift from a position within Rankin Inlet in early August and two sightings were made. The last, an apparent one by radar only, put it in a position (61°29'N 85°26'W) about 70 miles southwest of Coats Island on September 17.

Hachey (1935) on the basis of drift bottle recoveries from releases (Table II, Figure 17) on August 6, 7, 19, and September 5, 1930 concluded that "the general circulation of the surface waters of Hudson Bay would seem to be an anticlockwise one, with the outward movement of the waters from James Bay area holding close to the east coast of Hudson Bay". This circulation is consistent with the likely pattern of movement of the fresh water from run-off entering Hudson Bay, particularly in the area south from Churchill to James Bay and along the east shore, so that it is not necessary to postulate onshore transport due to wind in order to explain the recoveries. A noteworthy feature of the returns is that none were reported from any part of northwest Hudson Bay from about Churchill to southern Coats Island. This suggests that the surface movement here, relative to the release areas, was away from the region. A peculiarity of the surface circulation in many seasons relates to the accumulation of ice in the southwest portion. The accumulation is not entirely compatible with the anticlockwise movement inferred from the drift bottle recoveries; indeed Markham (1962) from ice observations was led to suggest that perhaps northerly movement occurs in July and August of that ice in the vicinity of Churchill. Such a movement would be compatible with the lack of drift bottle recoveries described, and supports the existence of an offshore movement of the surface water in the northwest region. Movements into the region from Roes Welcome Sound and from Hudson Strait through Fisher Strait could be expected and have been reported, and as well an upwelling of deeper water as described in the section on surface wind could be expected. That a southerly movement out of Roes Welcome Sound did not occur prior to July in 1961 is deduced in section 4.4 from features of the seasonal heat storage.

Later, in August, a relatively high heat storage is indicated (Fig. 18, a) for much of Hudson Bay. Locally some departure from this occurred, particularly to the northeast of Coats Island where an extremely high value was observed, and off the southern extremity of Southamption Island where an extremely low value was observed. In the southwest the values were estimated from data observed to about mid-August and, as would be anticipated, strongly reflect the influence of the ice cover as has been described. Heat budget calculations from the water adjacent to Churchill indicate that

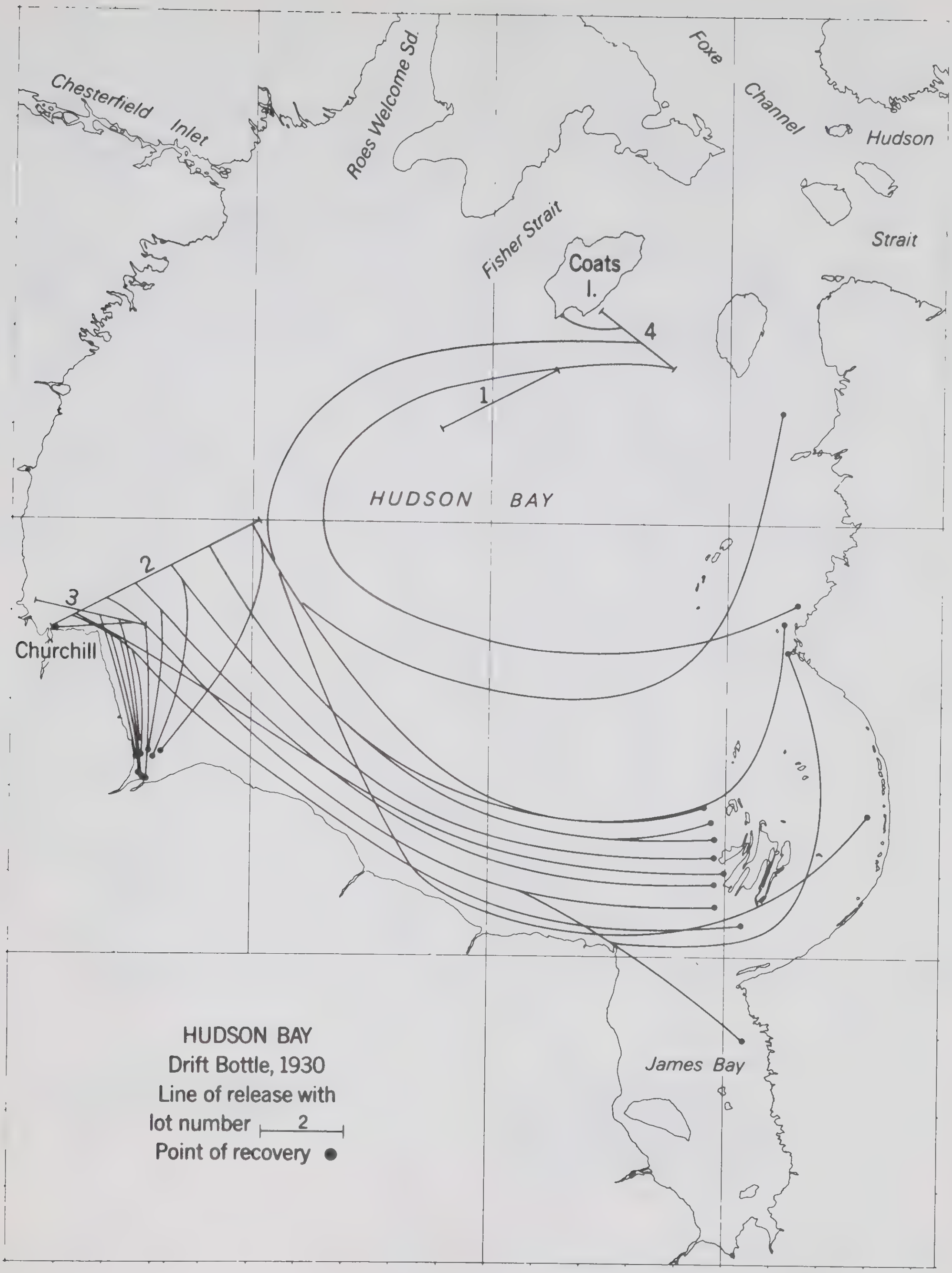


Figure 17. A re-presentation of drift bottle data of Hachey (1935) showing the line along which the bottles were released, the approximate point of recovery, and the inferred movement. The time of release of each lot is indicated in Table II.

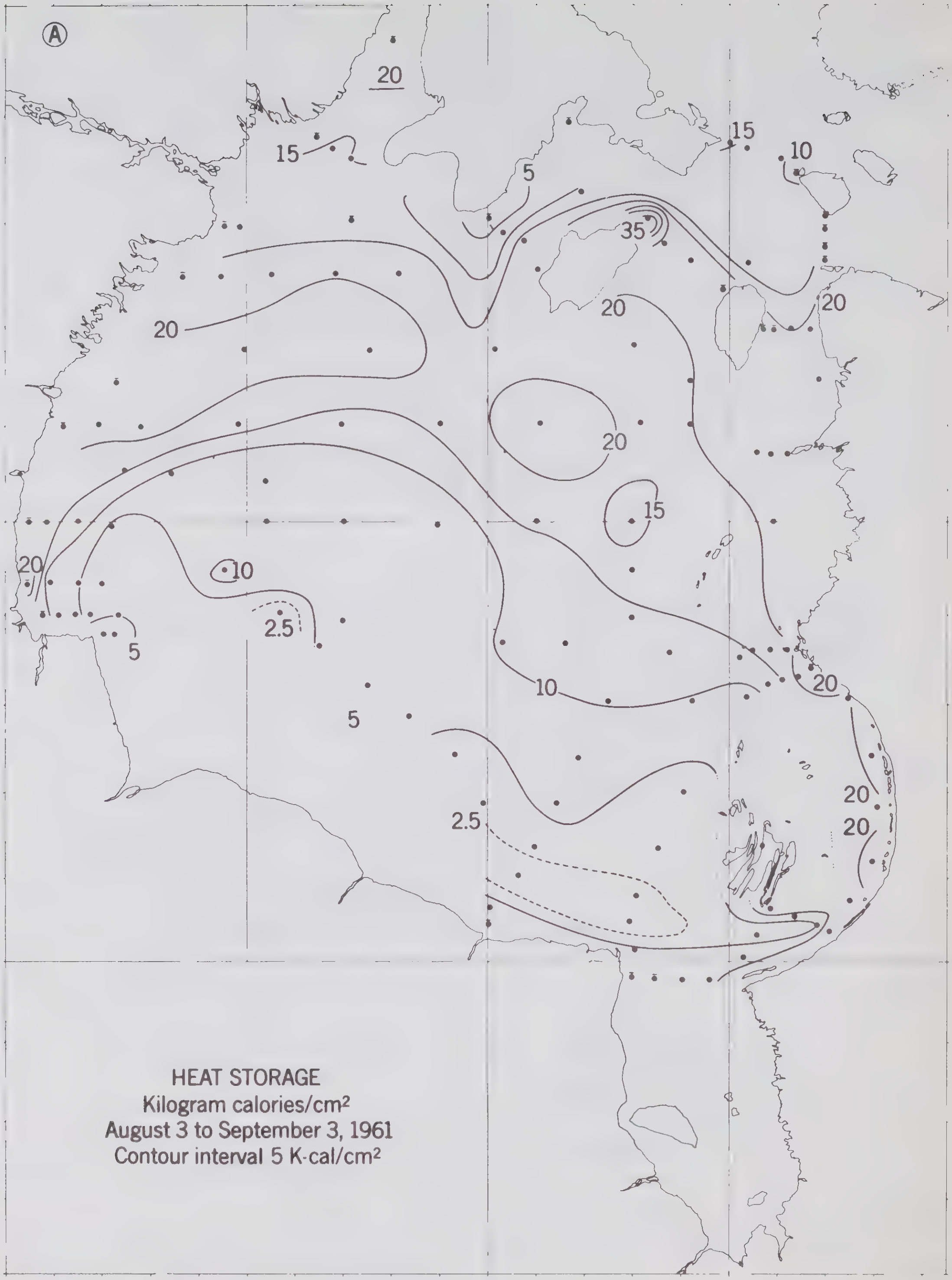
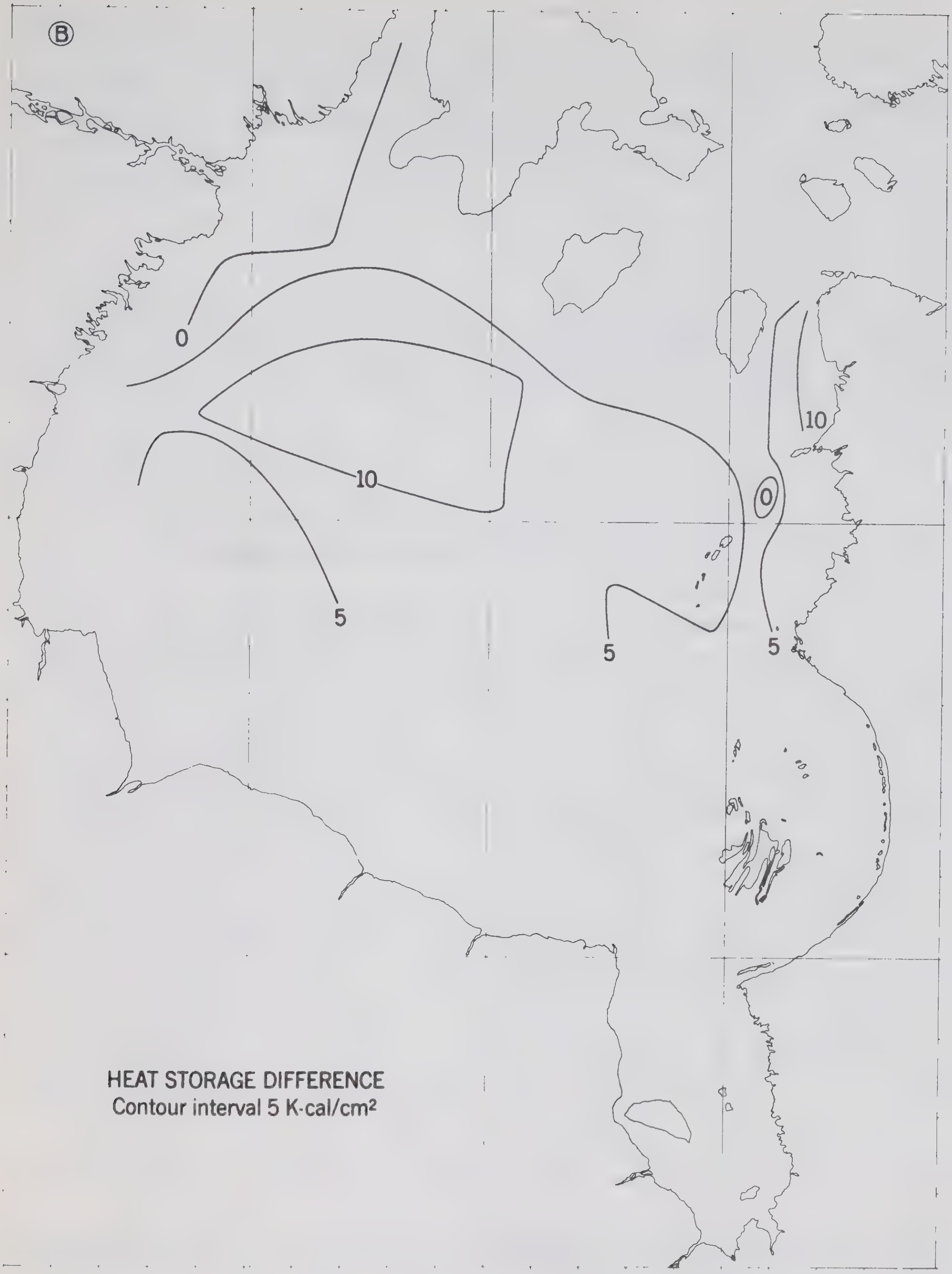


Figure 18. Features of the seasonal heat storage in kilogram calories per square centimeter. (a) Heat storage in August 1961. (b) Change of heat storage



during period July to August 1961 (the July distribution is discussed in section 4. 4).

in August 1961 the net of the radiative and flux terms to be about 5 kg cal/cm^2 into the water. Assuming this to be generally representative of the northern portion and in the absence of significant advection the August heat content should be 5 kg cal/cm^2 greater than that for July. The observed difference is shown in Fig. 18, b. Indicated is an area in the north central portion of average, i.e., 5 kg cal/cm^2, and surplus heat storage; deficit values are indicated close to the coast in the northwest and in the north, and in the section Ottawa Islands to Mansel Islands.

The deficit in the latter region could be due to advection from the south of relatively cold water recently influenced by ice cover and ice melt; the accumulation of ice in the southwest leads to such an explanation. The existence of the ice also leads to the conclusion that the heat surplus in the north central area must come about through an advection from the north rather than the south. An interpretation of the figure for the difference of heat content suggests the northwest area as a likely partial source. Both the drift bottle and heat storage data then suggest the northwest may be a region where in many seasons a departure from the simple anticlockwise movement of surface water occurs. As already inferred this could be due to wind.

The distribution of density at depth in the northwest in 1961 (Fig. 6, d) in which occurred a shoreward shallowing of sigma-t surface with an offshore trough is compatible with the existence of a surface outflow there due to wind. It does seem however that the configuration may be determined to an extent by the relative location of the isolated cold water and the tongue of warm water* (Fig. 6 a, b), so that it is probable that the distribution reflects as well the influence of the winter and summer processes which occur predominately in Hudson Strait. The inference constitutes one of the more interesting aspects of the physical oceanography of Hudson Bay and a comprehensive description would require considerably more data than now exist. Of related interest is the extent to which a re-circulation of the deep water occurs within Hudson Bay. Some evidence for this exists as mentioned earlier and as indicated in the distribution of dissolved oxygen on the sigma-t surface (Fig. 7), but this is only one possible interpretation of the data. In this, water of low oxygen content is shown as extending across the northern portion suggesting a movement from east to west. The distribution of depth on the density surface (Fig. 6, d) is relatively flat in this region, but on a deeper surface not shown the possibility of a movement from east to west across northern Hudson Bay merging with a movement out of Hudson Strait is better defined. On the surface a general orientation about a relatively shallow and centrally located region as in Figure 6, d is marked. Thus the existence of an anti-

*The incomplete coverage of the survey of September 1962 (Fig. 6, c) indicates the warm tongue to be mo.e extensive than observed in either July or August of 1961, and the position of the cold water to be to the southwest of that observed in 1961. These features are compatible with the visualized formation of water in Hudson Strait and the subsequent movement anticlockwise about Hudson Bay. It is possible to derive a speed for this movement (the position in each year and the estimated mean time of formation of the cold water indicates a movement of about 300 miles in about 5 months or about 2 miles/day or 4 cm/sec), but the data are not sufficient to allow acceptance of the derivation without more than the usual reservation.

clockwise movement at depth throughout the unrestricted part of Hudson Bay appears to occur. Some departure may be noted in the south where the surface is again relatively flat. This is true of the deeper surfaces which taken together suggest a restriction of water movement. This could be due to an influence of the central shoal (Fig. 7), and perhaps to an influence associated with the movement or flow at depth into James Bay.

The assumption that such an inflow occurs follows from a consideration of the influences likely to be significant there. An assessment of them for James Bay could proceed in the same manner as in this chapter for Hudson Bay, except that the data are even fewer. In this respect most of the southeastern portion of Hudson Bay is similar to James Bay. An obvious conclusion is that more data are required for the entire system.

4.7 Suggestions for future work

Because of the restricted period of navigation and consequent difficulty of data acquisition over much of the year, it would seem of more than the usual importance that the data requirement, in terms of number and location of stations and frequency of observation, be carefully established. The considerable influence of Hudson Strait needs elaboration, which in turn would lead to a program of observations, in that equally difficult area, although it appears that data now available have not been fully exploited. Information on the extent and location of open water, and the amount of ice cover at frequent intervals would be required, particularly in areas of intense mixing as Hudson Strait. More meteorological observations would likely be required as would information on the distribution and amount of fresh water from run-off. Time series observations, suitably located, would probably provide the most useful type of data; a consideration which applies to Hudson Strait as well as Hudson Bay. The data for Hudson Bay might have a more fundamental application for it seems possible that an increased general understanding of the processes significant to the growth and decay of the mixed layer could result. Stewart (1963) remarked on this with regard to the Arctic Ocean, but work there does not appear to have been undertaken. It is considered here that certain characteristics of Hudson Bay, arising from the degree of isolation from the main ocean and the atmosphere, indicate that it may be a particularly suitable location for such experiment.

5. SUMMARY

A review of source data in Hudson Bay to 1962 is provided. Distributions of salinity, temperature, sigma-t, and dissolved oxygen as observed during the navigation period of summer are described and the factors which lead to these distributions discussed. An attempt to assess these factors and their influence leads to considerable speculation which in turn provides the basis for a simple model of events and for a pattern of the circulation. The degree of speculation serves both to emphasize the paucity of data and to suggest the nature of future work. Hudson Bay may provide a suitable region for certain fundamental studies in oceanography.

6. ACKNOWLEDGEMENTS

Significant contributions to the work were made by A. R. Clarke, B. J. Layton, T. B. Moodie, and Miss. N. Pook during tenures as summer assistants. Mr. M. Isabelle prepared the line drawings, and with B. J. Kelly, assisted with data preparation. Miss. S. Chabaylo typed drafts of the manuscript.

Mr. R. Patenaude designed and mounted the stereograms of the Plate. Dr. G. L. Pickard read the manuscript and provided a critical review.

7. REFERENCES

Anonymous, undated, a. Monthly record of meteorological observations in Canada. Department of Transport, Canada. Meteorological Branch.

undated, b. General summary of hourly weather observations. Canada Department of Transport, Meteorological Branch.

1931, a. Sixty-third annual report. Marine Branch, Department of Marine and Fisheries. Canada.

1931, b. Sixty-fourth annual report. Department of Marine. Canada.

1931, c. Navigation conditions in Hudson Bay and Strait during season of navigation 1930. Department of Marine. Canada.

1932. Sixty-fifth annual report. Department of Marine. Canada.

1954. Labrador and Hudson Bay pilot. Can. Hydrog. Serv., Department of Mines & Technical Surveys.

1959. Pilot of Arctic Canada. Can. Hydrog. Serv., Department of Mines & Technical Surveys 2.

1960. Tidal and Oceanographic Survey Hudson Strait. August and September 1959. Canadian Hydrographic Service.

1961. Ice thickness data for Canadian stations. Met. Branch. Canada, Department of Transport, Cir 3537:1-65.

1964, a, b. Hudson Bay project - 1961. Data Record Series No. 1. Queen's Printer and Controller of Stationery Cat. No. N58-1/1964-1. Series No. 12. Cat. No. M58-1/1964-12.

1964, c. CSS "Acadia", 50 years of service. Marine Sciences Branch. Department of Mines & Technical Surveys. Canada.

1965, a. Oceanographic data catalogue, Canadian East Coast. Canadian Oceanographic Data Centre. Queen's Printer and Controller of Stationery Cat. No. M52-1965.

1965, b. Labrador and Hudson Bay Pilot. Canadian Hydrographic Service.

1966. Eastern Arctic and Hudson Bay - 1962. Data Record Series No. 4. Queen's Printer and Controller of Stationery (in press).

Anderson, Ernest R., 1952. Energy budget studies. Water loss investigations Lake Hefner Studies Tech. Rept., U. S. Geological Survey. Professional Paper 269, 71-117.

Archibald, D. C., M. N. Monsinger and T. B. Kilpatrick, 1962. Aerial ice observing and reconnaissance: the Hudson Bay route-1961. Meteorological Branch. Department of Transport. Canada. Cir-3710. Tec-421. pp. 1-69.

Bailey, W. B. and H. B. Hachey, 1951. An increasing Atlantic influence in Hudson Bay. Prov. Nova Scotia Inst. Science, XXII, Pt. 4.

Barber, F. G. and C. J. Glennie, 1964. On the oceanography of Hudson Bay: an atlas presentation of data obtained in 1961. Marine Sciences Br. Department of Mines & Technical Surveys, MS series No. 1.

Barber, F. G., 1965. Current observations in Fury and Hecla Strait. J. Fish. Res. Bd. Canada, 22(1).

(in review). On the oceanography of Hudson Bay. Submitted to the Journal of the Fisheries Research Board. Canada.

Bowen, I. S., 1926. The ratio of heat losses by conduction and by evaporation from any water surface. Phys. Rev., 27 : 779-787.

Campbell, N. J., 1958. The oceanography of Hudson Strait. Fish. Res. Bd. Canada, MS Rept. (Oceanogr. and Limnol.) No. 12, pp. 60.

1959. Some oceanographic features of northern Hudson Bay, Foxe Channel, and Hudson Strait. Ibid., No. 46, pp. 29.

1964. The origin of cold high-salinity water in Foxe Basin. J. Fish. Res. Bd. Canada, 21(1) : 45-55.

Campbell, N. J. and A. E. Collin, 1956. A preliminary report on some oceanographic features of Foxe Basin. Fish. Res. Bd. Canada, MS Rept. Biol. Sta., pp. 42.

1958. The discoloration of Foxe Basin ice. J. Fish. Res. Bd. Canada, 15(6) :1175-1188.

Cavadias, G. S., 1961. Evaporation application in watershed yield determinations. Proceedings of Hydrology Symposium No. 2-Evaporation. Water Resources Branch. Department of Northern Affairs and National Resources.

Coachman, L. K., 1963. Water masses of the Arctic. Proceedings of the Arctic Basin Symposium October 1962. Arctic Institute of North America.

Collin, A. E., 1958. An oceanographic study of Prince Regent Inlet, the Gulf of Boothia and adjacent waters. Fish. Res. Bd. Canada, MS Rept. (Oceanogr. and Limnol.) No. 13.

Donovan, F. P., 1957. A preliminary report on ice conditions on salt and fresh water bodies in the vicinity of Fort Churchill. DRNL Tech. Memo. No. 1/57.

Dunbar, M. J., 1951. Eastern Arctic Waters. Fish. Res. Bd. Canada. Bulletin No. 88.

1956. The "Calanus" expeditions in the Canadian Arctic, 1947 to 1955. Arctic Inst. N. A. 9(3) : 178-190.

1958. Physical oceanographic results of the "Calanus" expeditions in Ungava Bay, Frobisher Bay, Cumberland Sound, Hudson Strait, and northern Hudson Bay, 1949-1955. J. Fish. Res. Bd. Canada, 15(2) :155-201.

Dunbar, Moira, 1954. The pattern of ice distribution in Canadian Arctic seas. Trans. Roy. Soc. Canada, 48, Ser. III : 9-18.

Edgerton, Harold E., 1964. Application of photography to observations in the sea. Tenth Pacific Science Congress Series. Univ. Hawaii Press. John E. Tyler, editor.

Farquharson, W. I. and C. D. Sauer, 1960. Tidal and oceanographic survey of a section of Hudson Strait, August and September 1959. Preliminary MS report Department of Mines and Technical Surveys.

Flaherty, R. J., 1918. The Belcher Islands of Hudson Bay: their discovery and exploration. Geographical Review, 5(6) : 433-458.

Fofonoff, N. P., 1962. Physical properties of sea-water. The Sea Volume I, M. N. Hill, ed., Interscience Publishers.

Grainger, E. H., 1954. Station list of the "Calanus" expeditions, 1951-52, together with Frobisher Bay stations, 1948, 1950 and 1951, and Resolution Island stations, 1950. J. Fish. Res. Bd. Can., 11(1) : 98-105.

1959. The annual oceanographic cycle at Igloolik in the Canadian Arctic. 1. The zooplankton and physical and chemical observations. J. Fish. Res. Bd. Canada, 16(4) : 453-501.

1960. Some physical oceanographic features of southeast Hudson Bay and James Bay. Fish. Res. Bd. Canada, MS (Oceanog. and Limnolog.) No. 71.

1963. Capepods of the genus Calanus as indicators of eastern Canadian waters. Royal Soc. Canada. Special Publications No. 5. Univ. Toronto Press.

Grainger, E. H. and M. J. Dunbar, 1956. Station list of the "Calanus" expeditions, 1953-4. J. Fish. Res. Bd. Canada, 13(1) : 41-45.

Grainger, E. H. and J. G. Hunter, 1959. Station list of the 1955-58 field investigations of the Arctic Unit of the Fisheries Research Board of Canada. J. Fish. Res. Bd. Canada, 16(4) : 403-420.

Hachey, H. B. , 1931, a. The general hydrography and hydrodynamics of the waters of the Hudson Bay region. Contrib. Canad. Biol. Fisher. 7(9) : 93-118.

1931, b. Report on the Hudson Bay fisheries expedition of 1930. Contrib. Canad. Biol. Fisher., N. S. 6(23), 3-9.

1935. The circulation of Hudson Bay water as indicated by drift bottles. Science 82 (2125) 275-6.

1954. They hydrography of Hudson Bay. Proc. Roy. Soc. Canada, 48, Ser III: 19-23.

Hare, F. K. and M. R. Montgomery, 1949. Ice, open water, and winter climate in the eastern Arctic of North America. Arctic 2:2: 79-89 (Part 1) Sept. 1949; 2:3 :149-164 (Part 2) Dec. 1949.

Hood, Peter, 1964. Sea magnetometer reconnaissance of Hudson Bay. Geophysics 29(6): 916-921.

1966. Geophysical reconnaissance of Hudson Bay. Geological Survey Canada Paper (in press).

Johnson, Lionel, 1965. The salinity of Baker Lake, N. W. T. , Canada. J. Fish. Res. Bd. Canada 22(1) : 239-241.

Laevastu, T. , 1960. Factors affecting the temperature of the surface layer of the sea. Soc. Scient. Fennica, Comment. Physico-Mathem. 25(1) : pp. 1-136.

Lamont, A. H. , 1949. Ice conditions over Hudson Bay and related weather phenomena. Am. Meteorol. Soc. Bull. 30. 8.

Leslie, R. J. , 1963. Foraminiferal study of a cross-section of Hudson Bay, Canada. Geological Survey of Canada, Paper 63-16: 1-28.

1964. Sedimentology of Hudson Bay District of Keewatin. Geological Survey of Canada, Paper 63-48: 1-31.

1965. Ecology and paleoecology of Hudson Bay foraminifera. Unpublished Manuscript. Bedford Institute of Oceanography Report 65-6.

Leslie, R. J. and B. R. Pelletier, 1965. Bedrock geology beneath Hudson Bay as interpreted from submarine physiography. Unpublished Manuscript. Bedford Institute of Oceanog. 65-12.

Mackay, G. A., 1952. The effect of protracted spring thaws on ice conditions in Hudson Bay. Bull. Amer. Met. Soc. 33(3).

Mackay, D. K. and J. R. Mackay, 1965. Historical records of freeze-up and break-up on Churchill and Hayes Rivers. Geog. Br., Department of Mines & Technical Surveys, Geog. Bulletin 7(1) : 7-16.

Markham, W. E., 1962. Summer break-up patterns in the Canadian Arctic. Meteorol. Branch. Department of Transport, Canada, Cir-3586. Tec-389: 1-8.

Mateer, C. L., 1955, a. Average insolation in Canada during cloudless days. Canadian J. Technol., 33 : 12-32.

1955, b. A preliminary estimate of the average insolation in Canada. Can. Jour. Agricultural Science 35 : 579-594.

Miyake, Y., 1939. Chemical studies of the western Pacific Ocean, 3, Freezing point, osmotic pressure, boiling point, and vapour pressure of sea water, J. Chem. Soc. Japan, 14, 58-62.

Montgomery, R. B., 1938. Circulation in upper layers of southern North Atlantic deduced with use of isentropic analysis. Pap. Phys. Oceanogr. Meteor., 6(2) : 1-55.

Pickard, G. L., 1964. Descriptive physical oceanography, an introduction. Pergamon Press Ltd., Oxford.

Smith, Edward H., 1931. The "Marion" expedition to Davis Strait and Baffin Bay. U. S. Coast Guard Bull., No. 19. Pt. 3, pp. 216.

Smith, E. H., F. M. Soule, and O. Mosby, 1937. The "Marion" and "General Greene" expeditions to Davis Strait and Labrador Sea. U. S. Treasury Department, Coast Guard Bull., No. 19(2).

Stewart, R. W., 1963. Some aspects of turbulence in the Arctic. Proceedings of the Arctic Basin Symposium October 1962. Arctic Institute of North America.

Sverdrup, H. V., M. W. Johnson, and R. H. Fleming, 1942. The oceans. Prentic-Hall, New York.

Swanson, L. M., 1961. Ice conditions in the Churchill area for the winter of 1960-61. Defence Res. Board, Canada. DRNL Note No. 8/61.

Thompson, Thomas G. Physical properties of sea water, Physics of the Earth, v. 5, Oceanography, pp. 63-94, Nat. Res. Counc. Bull. No. 85, Washington, D. C.

Tully, John P., 1958. On structure, entrainment, and transport in estuarine embayments. J. Marine Res. 17:523-535.

Untersteiner, N. and R. Sommerfeld, 1964. Supercooled water and the bottom topography of floating ice. J. Geophysical Research. 69(6) : 1057-1062.

Williams, G. P., 1961. Evaporation from water, snow and ice. Proceedings of Hydrology Symp., No. 2, Toronto : 31-47.